Name & CWID ______________________________

PH 253 Final Exam

Instructions

1. Solve six of the problems below. All problems have equal weight.
2. Clearly mark your which problems you have chosen.
3. Do your work on separate sheets. Staple them to this exam paper when you are finished.
4. You are allowed 2 sheet of standard 8.5x11 in paper and a calculator.

1. An interstellar space probe is moving at a constant speed relative to earth of $0.85c$ toward a distant planet. Its radioisotope generators have enough energy to keep its data transmitter active continuously for 12 years, as measured in their own reference frame.

(a) How long do the generators last as measured from earth?
(b) How far is the probe from earth when the generators fail, as measured from earth?
(c) How far is the probe from earth when the generators fail, *as measured by its built-in trip odometer*?

2. A Σ^0 particle at rest decays to a Λ^0 particle and a photon. Determine the energy of the released photon, given that the Σ^0 has rest energy $m_\Sigma c^2 = 1192\,\text{MeV}$ and the Λ^0 has rest energy $m_{\Lambda^0} c^2 = 1116\,\text{MeV}$.

3. The state of a free particle is described by the following wave function

$$\psi(x) = \begin{cases} 0 & x < -b \\ A & -b \leqslant x \leqslant 7b \\ 0 & x > 7b \end{cases} \tag{1}$$

(a) Determine the normalization constant A.
(b) What is the probability of finding the particle in the interval $[0, b]$?
(c) Determine $\langle x \rangle$ and $\langle x^2 \rangle$ for this state.
(d) Find the uncertainty in position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

4. A phenomenological expression for the potential energy of a bond as a function of spacing is given by

$$U(r) = \frac{A}{r^n} - \frac{B}{r^m} \tag{2}$$

For a stable bond, $m < n$. Show that the molecule will break up when the atoms are pulled apart to a distance

$$r_b = \left(\frac{n+1}{m+1}\right)^{1/(n-m)} r_o \tag{3}$$

where r_o is the equilibrium spacing between the atoms. Be sure to note your criteria for breaking used to derive the above result.

5. The speed of light in still water is c/n, where n is the index of refraction, approximately $n = 4/3$ for water. Fizeau, in 1851, found that the speed (relative to the laboratory) of light in water moving at speed V (relative to the laboratory) could be expressed as

$$u = \frac{c}{n} + kV \tag{4}$$

where the "dragging coefficient" was measured by him to be $k \approx 0.44$. Determine the value of k predicted by the Lorentz velocity transformations. Note $(1+x)^{-1} \approx 1 - x$ for $x \ll 1$.

6. In an experiment to find the value of h, light at wavelengths 218 and 431 nm were shone on a clean sodium surface. The potentials that stopped the fastest photoelectrons were 5.69 and 0.59 V, respectively. What values of h and W, the sodium work function, are deduced?

7. An electron initially moving at constant speed v is brought to rest with uniform deceleration a lasting for a time $t = v/a$. Compare the electromagnetic energy radiated during this deceleration with the electron's initial kinetic energy. Express the ratio in terms of two lengths, the distance light travels in time t and the classical electron radius $r_e = e^2/4\pi\epsilon_o mc^2$.

8. An electron in a helium atom is in a state described by the (normalized) wave function

$$\psi = \frac{4}{\sqrt{2\pi}\,(a_o)^{3/2}} e^{-2r/a_o} \tag{5}$$

where a_o is the Bohr radius.

(a) What is the *most probable* value of r?
(b) What is the energy E of the electron in this state? *Hint: use Schrödinger's equation.*

9. *Splitting of Hydrogen lines.* The electron's intrinsic magnetic moment $\vec{\mu}_s$ and intrinsic spin angular momentum $\vec{S}$ are proportional to each other; their relationship can be written as

$$\vec{\mu}_s = -g_s \frac{e}{2m}\vec{S} = -g\mu_b \vec{S} \tag{6}$$

with $g_s \approx 2$. The energy of the electron in a effective magnetic field $\vec{B}$ is $E = -\vec{\mu}_s \cdot \vec{B}$.

In hydrogen, transitions occur between two spin-orbit-split 2p states and a single 1s state, leading to two emission lines. If the emission wavelength in the absence of spin-orbit coupling is 656.47 nm, and the spin-orbit splitting is 0.016 nm, estimate the strength of the effective magnetic field produced by the electron's orbital motion (i.e., the effective field due to the spin-orbit interaction) which results in this wavelength difference.

10. *Multiplicity of atomic magnetic moments.* Calculate the magnetic moments that are possible for the

$n=4$ level of Hydrogen, making use of the quantization of angular momentum. You may neglect the existence of spin. Compare this with the Bohr prediction for $n=4$.

11. *Transitions in a magnetic field.* Transitions occur in an atom between $l = 2$ and $l = 1$ states in a magnetic field of 2.0 T, obeying the selection rules $\Delta m_l = 0, \pm 1$. If the wavelength before the field was turned on was 680.0 nm, determine the wavelengths that are observed. You may find the following relationship from last week's homework useful:

$$|\Delta\lambda| = \frac{\lambda^2 \Delta E}{hc} \tag{7}$$

Recall that the Zeeman effect changes the energy of a single-electron atom in a magnetic field by

$$\Delta E = m_l \left(\frac{e\hbar}{2m_e}\right) B \qquad \text{with} \qquad m_l = -l, -(l-1), \ldots, 0, \ldots, l-1, l \tag{8}$$

For convenience, note that $e\hbar/2m_e = \mu_B \approx 57.9\,\mu\text{eV/T}$, and neglect the existence of spin. See also: Pfeffer & Nir 3.2.2

12. Explain, appealing to band theory, why conductors tend to be opaque and insulators transparent. Your answer should include sketches.

Constants:

$$N_A = 6.022\times10^{23}\ \text{things/mol}$$
$$k_e \equiv 1/4\pi\epsilon_o = 8.98755\times10^9\ \mathrm{N\cdot m^2\cdot C^{-2}}$$
$$\epsilon_o = 8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}$$
$$\mu_o \equiv 4\pi\times10^{-7}\ \mathrm{T\cdot m/A}$$
$$e = 1.60218\times10^{-19}\ \mathrm{C}$$
$$h = 6.6261\times10^{-34}\ \mathrm{J\cdot s} = 4.1357\times10^{-15}\ \mathrm{eV\cdot s}$$
$$\hbar = \frac{h}{2\pi} \qquad hc = 1239.84\ \mathrm{eV\cdot nm}$$
$$k_B = 1.38065\times10^{-23}\ \mathrm{J\cdot K^{-1}} = 8.6173\times10^{-5}\ \mathrm{eV\cdot K^{-1}}$$
$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.99792\times10^8\ \mathrm{m/s}$$
$$m_e = 9.10938\times10^{-31}\ \mathrm{kg} \qquad m_e c^2 = 510.998\ \mathrm{keV}$$
$$m_p = 1.67262\times10^{-27}\ \mathrm{kg} \qquad m_p c^2 = 938.272\ \mathrm{MeV}$$
$$m_n = 1.67493\times10^{-27}\ \mathrm{kg} \qquad m_n c^2 = 939.565\ \mathrm{MeV}$$

Schrödinger

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + V(x)\Psi$$
$$E\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi$$
$$\int_{-\infty}^{\infty}|\psi(x)|^2\,dx = 1 \qquad P(\text{in }[x, x+dx]) = |\psi(x)|^2 \qquad \text{1D}$$
$$\int_0^{\infty}|\psi(r)|^2\,4\pi r^2\,dr = 1 \qquad P(\text{in }[r, r+dr]) = 4\pi r^2|\psi(r)|^2 \qquad \text{3D}$$
$$\langle x^n\rangle = \int_{-\infty}^{\infty}x^n P(x)\,dx \qquad \text{1D} \qquad \langle r^n\rangle = \int_0^{\infty}r^n P(r)\,dr \qquad \text{3D}$$
$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

Basic Equations:

$$\vec{F}_{\text{net}} = m\vec{a}\ \ \text{Newton's Second Law}$$
$$\vec{F}_{\text{centr}} = -\frac{mv^2}{r}\,\hat{\mathbf{r}}\ \ \text{Centripetal}$$
$$\vec{F}_{12} = k_e\frac{q_1q_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} = q_2\vec{E}_1 \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$
$$\vec{E}_1 = \vec{F}_{12}/q_2 = k_e\frac{q_1}{r_{12}^2}\,\hat{\mathbf{r}}_{12}$$
$$\vec{F}_B = q\vec{v}\times\vec{B}$$
$$0 = ax^2 + bx^2 + c \Longrightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Oscillators

$$E = \left(n+\frac{1}{2}\right)hf$$
$$E = \frac{1}{2}kA^2 = \frac{1}{2}\omega^2 mA^2 = 2\pi^2 mf^2A^2$$
$$\omega = 2\pi f = \sqrt{k/m}$$

Approximations, $x \ll 1$

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n+1)x^2 \qquad \tan x \approx x + \frac{1}{3}x^3$$
$$e^x \approx 1 + x + \frac{1}{2}x \qquad \sin x \approx x - \frac{1}{6}x^3 \qquad \cos x \approx 1 - \frac{1}{2}x^2$$

Misc Quantum

$$E = hf \qquad p = h/\lambda = E/c \qquad \lambda f = c \qquad \text{photons}$$
$$\lambda_f - \lambda_i = \frac{h}{m_e c}(1-\cos\theta)$$
$$\lambda = \frac{h}{|\vec{p}|} = \frac{h}{\gamma mv} \approx \frac{h}{mv}$$
$$\Delta x\Delta p \geqslant \frac{h}{4\pi} \qquad \Delta E\Delta t \geqslant \frac{h}{4\pi}$$
$$eV_{\text{stopping}} = KE_{\text{electron}} = hf - \varphi = hf - W$$

Bohr

$$E_n = -13.6\ \mathrm{eV}/n^2 \qquad \text{Hydrogen}$$
$$E_n = -13.6\ \mathrm{eV}\left(Z^2/n^2\right) \qquad Z\text{ protons, }1\ e^-$$
$$\Delta E = -13.6\ \mathrm{eV}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = hf$$
$$L = mvr = n\hbar$$
$$v^2 = \frac{n^2\hbar^2}{m_e^2r^2} = \frac{k_e e^2}{m_e r}$$

Quantum Numbers

$$l = 0, 1, 2, \ldots, (n-1) \qquad L^2 = l(l+1)\hbar^2$$
$$m_l = -l, (-l+1), \ldots, l \qquad L_z = m_l\hbar$$
$$m_s = -\pm\frac{1}{2} \qquad S_z = m_s\hbar \qquad S^2 = s(s+1)\hbar^2$$
$$\text{dipole transitions: } \Delta l = \pm1, \Delta m_l = 0, \pm1, \Delta m_s = 0$$
$$\mu_{sz} = \pm\mu_B$$
$$\vec{\mu}_s = 2\vec{S}\,\mu_B$$
$$E_\mu = -\vec{\mu}\cdot\vec{B}$$
$$J^2 = j(j+1)\hbar^2 \qquad j = l\pm\frac{1}{2}$$
$$J_z = m_j\hbar \qquad m_j = -j, (-j+1), \ldots, j$$

Calculus of possible utility:

$$\int\frac{1}{x}\,dx = \ln x + c$$
$$\int u\,dv = uv - \int v\,du$$
$$\int\sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int\cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$$
$$\int_0^{\infty}x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$
$$\int_0^{\infty}x^2e^{-ax^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$
$$\int_{-\infty}^{\infty}x^3e^{-ax^2}\,dx = \int_{-\infty}^{\infty}xe^{-ax^2}\,dx = 0$$
$$\int_0^{\infty}x^4e^{-ax^2}\,dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$

E & M

$$\vec{F}_{12} = k_e\frac{q_1q_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} = q_2\vec{E}_1 \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$
$$\vec{E}_1 = \vec{F}_{12}/q_2 = k_e\frac{q_1}{r_{12}^2}\,\hat{\mathbf{r}}_{12}$$
$$\vec{F}_B = q\vec{v}\times\vec{B}$$

Blackbody

$$E_{tot} = \sigma T^4 \qquad \sigma = 5.672 \times 10^{-8}\,\mathrm{W \cdot m^{-2} \cdot K^{-4}}$$

$$T\lambda_{max} = 0.29 \times 10^{-2}\,\mathrm{m \cdot K} \qquad \text{Wien}$$

$$E_{quantum} = hf$$

$$E_{oscillator} = hf / \left(e^{hf/k_B T} - 1\right)$$

$$I(\lambda, T) = \frac{(\text{const})}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1}$$

$$I(f, t) = (\text{const})\, f^3 \left[e^{\frac{hf}{k_b T}} - 1\right]^{-1}$$

Relativity

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta t'_{moving} = \gamma \Delta t_{stationary} = \gamma \Delta t_p$$

$$L'_{moving} = \frac{L_{stationary}}{\gamma} = \frac{L_p}{\gamma}$$

$$x' = \gamma\,(x - vt)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$v_{obj} = \frac{v + v'_{obj}}{1 + \frac{vv'_{obj}}{c^2}} \qquad v'_{obj} = \frac{v_{obj} - v}{1 - \frac{vv_{obj}}{c^2}}$$

$$KE = (\gamma - 1)mc^2 = \sqrt{m^2c^4 + c^2p^2} - mc^2$$

$$E_{rest} = mc^2$$

$$p = \gamma m v$$

$$E^2 = p^2c^2 + m^2c^4 = \left(\gamma mc^2\right)^2$$

Vectors:

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2} \quad \text{magnitude} \qquad \theta = \tan^{-1}\left[\frac{F_y}{F_x}\right] \quad \text{direction}$$

$$\hat{r} = \vec{r}/|\vec{r}| \quad \text{construct any unit vector}$$

$$\text{let} \quad \vec{a} = a_x\,\hat{x} + a_y\,\hat{y} + a_z\,\hat{z} \quad \text{and} \quad \vec{b} = b_x\,\hat{x} + b_y\,\hat{y} + b_z\,\hat{z}$$

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z = \sum_{i=1}^{n} a_i b_i = |\vec{a}||\vec{b}|\cos\theta$$

$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = (a_y b_z - a_z b_y)\,\hat{x} + (a_z b_x - a_x b_z)\,\hat{y} + (a_x b_y - a_y b_x)\,\hat{z}$$

Units

$$\begin{aligned}
1\,\mathrm{T \cdot m/A} &= 1\,\mathrm{N/A^2} \\
1\,\mathrm{T \cdot m^2} &= 1\,\mathrm{V \cdot s} \\
1\,\mathrm{T} &= 1\,\mathrm{kg/A \cdot s^2} \\
1\,\mathrm{eV} &= 1.6 \times 10^{-19}\,\mathrm{J} \\
1\,\mathrm{J} &= 1\,\mathrm{N \cdot m} = 1\,\mathrm{kg \cdot m^2/s^2} \\
1\,\mathrm{N} &= 1\,\mathrm{kg \cdot m/s^2} \\
1\,\mathrm{W} &= 1\,\mathrm{J/s} = 1\,\mathrm{kg \cdot m^2/s^3} \\
1\,\mathrm{F} &= 1\,\mathrm{C/V} \qquad 1\,\mathrm{C} = 1\,\mathrm{A/s} \\
1\,\mathrm{N/C} &= 1\,\mathrm{V/m}
\end{aligned}$$