PH 253 Exam III

Solve 4 of the 6 problems below. All problems have equal weight.

1. A long time ago, in a galaxy far, far away, electric charge had not yet been invented, and atoms were held together by gravitational forces. Find the equivalent of the Bohr radius for a gravitationally bound atom **symbolically**, and then make a numerical estimate.

Note that the Bohr radius for an electrically-bound atom is $a_o = 4\pi\epsilon_o\hbar^2/m_e e^2$. You should be able to find the correct result by comparing the electric and gravitational potential energies below and making the appropriate substitutions.

$$U_e = -\frac{e^2}{4\pi\epsilon_o r} \qquad U_g = -\frac{Gm_p m_e}{r} \tag{1}$$

here m_e is the electron mass, m_p the proton mass, and $G = 6.67\times10^{-11}\,\mathrm{N\,m^2/kg^2}$ is the gravitational constant. Dimensional analysis may be a good double check.

2. *Energetics of diatomic systems* An expression for the potential energy of two neutral atoms as a function of their separation r is given by the *Morse potential*,

$$PE = P_o\left[1 - e^{-a(r-r_o)}\right]^2 \tag{2}$$

Find the equilibrium spacing and dissociation energy. *Note:* Equilibrium is characterized by $dU/dr = 0$. The dissociation energy is defined as the amount of energy required to take the system from equilibrium at $r = r_o$ to complete breakup for $r \to \infty$

3. A hydrogen atom is in the $n=6$ state. **(a)** Counting *all* the possible paths, how many different photon energies can be emitted if the atom ends up in the ground state ($n=1$)? **(b)** Suppose only $\Delta n = 1$ transitions were allowed. How many different photon energies would be emitted?

4. A hydrogen atom is in the $n=2$, $l=1$, $m_l=0$ state, with a radial wave function

$$R(r) = \frac{1}{\sqrt{3}\,(2a_o)^{3/2}}\frac{r}{a_o}e^{-r/2a_o} \tag{3}$$

(a) What is the most probable radius to find the electron? **(b)** What is the energy of the electron in this state? The radial version of Schrödinger's equation is below, should you need it.

$$-\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \left(-\frac{e^2}{4\pi\epsilon_o r} + \frac{l\,(l+1)\,\hbar^2}{2mr^2}\right)R(r) = ER(r) \tag{4}$$

5. Compute the change in wavelength of the $2p \to 1s$ photon when a hydrogen atom is placed in a magnetic field of 2.00 T (neglect spin). You may find the following relationship useful:

$$|\Delta\lambda| = \left|\frac{d\lambda}{dE}\right|\Delta E = \frac{hc}{E^2}\Delta E = \frac{\lambda^2}{hc}\Delta E \tag{5}$$

For convenience, also note that $e\hbar/2m_e = \mu_B \approx 57.9\,\mu\,\text{eV/T}$.

6. Explain why each of the following sets of quantum numbers (n, l, m_l, m_s) is not permitted for hydrogen:

$$(3, 3, -1, +\frac{1}{2})$$
$$(7, 3, +4, -\frac{1}{2})$$
$$(4, 1, +1, \frac{3}{2})$$
$$(3, -2, +2, +\frac{1}{2})$$

Constants:

$$k_e \equiv 1/4\pi\epsilon_o = 8.98755\times10^9\ \mathrm{N\cdot m^2\cdot C^{-2}}$$
$$\epsilon_o = 8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}$$
$$e = 1.60218\times10^{-19}\ \mathrm{C}$$
$$h = 6.6261\times10^{-34}\ \mathrm{J\cdot s} = 4.1357\times10^{-15}\ \mathrm{eV\cdot s}$$
$$\hbar = \frac{h}{2\pi} \qquad hc = 1239.84\,\mathrm{eV\cdot nm}$$
$$\mu_B = e\hbar/2m_e = 9.274009\ \mathrm{J/T} \approx 57.9\,\mu\mathrm{eV/T}$$
$$k_B = 1.38065\times10^{-23}\ \mathrm{J\cdot K^{-1}} = 8.6173\times10^{-5}\ \mathrm{eV\cdot K^{-1}}$$
$$c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.99792\times10^8\ \mathrm{m/s}$$
$$m_e = 9.10938\times10^{-31}\,\mathrm{kg} \qquad m_ec^2 = 510.998\,\mathrm{keV}$$
$$m_p = 1.67262\times10^{-27}\,\mathrm{kg} \qquad m_pc^2 = 938.272\,\mathrm{MeV}$$

Schrödinger

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + V(x)\Psi \quad \text{time-dep, 1D}$$
$$E\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi \quad \text{time-indep, 1D}$$
$$\int_{-\infty}^{\infty}|\psi(x)|^2\,dx = 1 \qquad \text{1D}$$

$$P(\text{in }[x,x+dx]) = |\psi(x)|^2 \qquad \text{1D}$$
$$\int_0^{\infty}|\psi(r)|^2\,4\pi r^2\,dr = 1 \qquad \text{3D}$$
$$P(\text{in }[r,r+dr]) = 4\pi r^2|\psi(r)|^2 \qquad \text{3D}$$
$$\int_0^{\infty}|R(r)|^2\,r^2\,dr = 1 \qquad \text{radial only}$$
$$P(\text{in }[r,r+dr]) = r^2|\psi(r)|^2 \qquad \text{radial only}$$
$$\langle x^n\rangle = \int_{-\infty}^{\infty}x^n P(x)\,dx \qquad \text{1D}$$
$$\langle r^n\rangle = \int_0^{\infty}r^n P(r)\,dr \qquad \text{3D}$$
$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

Basic Equations:

$$\vec{\mathbf{F}}_{\text{net}} = m\vec{\mathbf{a}} \ \text{ Newton's Second Law}$$
$$\vec{\mathbf{F}}_{\text{centr}} = -\frac{mv^2}{r}\hat{\mathbf{r}} \ \text{ Centripetal}$$
$$\vec{\mathbf{F}}_{12} = k_e\frac{q_1q_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} = q_2\vec{\mathbf{E}}_1 \qquad \vec{\mathbf{r}}_{12} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$$
$$\vec{\mathbf{E}}_1 = \vec{\mathbf{F}}_{12}/q_2 = k_e\frac{q_1}{r_{12}^2}\,\hat{\mathbf{r}}_{12}$$
$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}}\times\vec{\mathbf{B}}$$
$$0 = ax^2 + bx^2 + c \Longrightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Oscillators

$$E = \left(n+\frac{1}{2}\right)hf$$
$$E = \frac{1}{2}kA^2 = \frac{1}{2}\omega^2mA^2 = 2\pi^2mf^2A^2$$
$$\omega = 2\pi f = \sqrt{k/m}$$

Approximations, $x \ll 1$

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n+1)x^2 \qquad \tan x \approx x + \frac{1}{3}x^3$$
$$e^x \approx 1 + x + \frac{1}{2}x \qquad \sin x \approx x - \frac{1}{6}x^3 \qquad \cos x \approx 1 - \frac{1}{2}x^2$$

E & M

$$\vec{\mathbf{F}}_{12} = k_e\frac{q_1q_2}{r_{12}^2}\,\hat{\mathbf{r}}_{12} = q_2\vec{\mathbf{E}}_1 \qquad \vec{\mathbf{r}}_{12} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$$
$$\vec{\mathbf{E}}_1 = \vec{\mathbf{F}}_{12}/q_2 = k_e\frac{q_1}{r_{12}^2}\,\hat{\mathbf{r}}_{12}$$
$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}}\times\vec{\mathbf{B}}$$

Misc Quantum/Relativity

$$E^2 = p^2c^2 + m^2c^4 = \left(\gamma mc^2\right)^2$$
$$E = hf \qquad p = h/\lambda = E/c \qquad \lambda f = c \qquad \text{photons}$$
$$\lambda_f - \lambda_i = \frac{h}{m_ec}(1-\cos\theta) \quad \text{Compton}$$
$$\lambda = \frac{h}{|\vec{\mathbf{p}}|} = \frac{h}{\gamma mv} \approx \frac{h}{mv}$$
$$\Delta x\Delta p \geq \frac{h}{4\pi} \qquad \Delta E\Delta t \geq \frac{h}{4\pi}$$
$$eV_{\text{stopping}} = KE_{\text{electron}} = hf - \varphi = hf - W$$

Bohr

$$E_n = -13.6\,\mathrm{eV}\left(Z^2/n^2\right) \qquad Z \text{ protons, 1 e}^-$$
$$\Delta E = -13.6\,\mathrm{eV}\left(\frac{1}{n_i^2} - \frac{1}{n_f^2}\right) = hf$$
$$L = mvr = n\hbar$$
$$r_n = \frac{4\pi\epsilon_o\hbar^2}{m_ee^2}n^2 = a_on^2$$
$$v^2 = \frac{n^2\hbar^2}{m_e^2r^2} = \frac{k_ee^2}{m_er}$$

Quantum Numbers

$$l = 0,1,2,\ldots,(n-1) \qquad L^2 = l(l+1)\hbar^2$$
$$m_l = -l,(-l+1),\ldots,l \qquad L_z = m_l\hbar$$
$$m_s = -\pm\frac{1}{2} \qquad S_z = m_s\hbar$$

dipole transitions: $\Delta l = \pm1, \Delta m_l = 0,\pm1, \Delta m_s = 0$

Calculus of possible utility:

$$\int u\,dv = uv - \int v\,du$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int_0^{\infty}x^ne^{-ax}\,dx = \frac{n!}{a^{n+1}}$$
$$\int_0^{\infty}x^2e^{-ax^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \qquad \int_0^{\infty}x^4e^{-ax^2}\,dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$
$$\int_{-\infty}^{\infty}x^3e^{-ax^2}\,dx = \int_{-\infty}^{\infty}xe^{-ax^2}\,dx = 0$$