Exam 2 practice problems

1. Determine the maximum scattering angle in a Compton experiment for which the scattered photon can produce a positron-electron pair. *Hint:* twice the electron's rest energy m_ec^2 is required of the incident photon for pair production.

Solution: In order to produce a positron-electron pair, enough energy must be present to supply the rest energy of a positron and an electron. Both have the same mass, m_e, so that means the incident photon must supply at least $2m_ec^2$ worth of enery. The threshold wavelength is thus

$$hf = \frac{hc}{\lambda_{th}} = 2m_ec^2 \qquad \text{or} \qquad \frac{h}{m_ec} = 2\lambda_{th} \tag{1}$$

Any wavelengths *above* this value cannot result in pair production. Substituting this into the Compton formula,

$$\lambda_f = \lambda_i + \frac{h}{m_ec}\left(1-\cos\theta\right) = \lambda_i + 2\lambda_{th}\left(1-\cos\theta\right) \tag{2}$$

The right side of the expression is the sum of two positive-definite terms. The right side can be at most λ_{th} - if $\lambda_f > \lambda_{th}$, pair production cannot occur. Even if λ_i is arbitrarily small, if

$$2\lambda_{th}\left(1-\cos\theta\right) \geq \lambda_{th} \tag{3}$$

then $\lambda_f > \lambda_{th}$. This leads us to a condition on the threshod angle θ_{th}

$$1-\cos\theta_{th} = \frac{1}{2} \qquad \text{or} \qquad \theta_{th} = 60^\circ \tag{4}$$

Alternate Solution

All this means is that the exiting (scattered) photon must have an energy of at least $2mc^2$. In terms of the dimensionless photon energies $\alpha_i = hf_i/mc^2$, $\alpha_f = hf_f/mc^2$, the Compton equation reads

$$\frac{1}{\alpha_i} = \frac{1}{\alpha_f} - \left(1-\cos\theta\right) \tag{5}$$

If the exiting photon energy is $hf_f = 2mc^2$, this means $\alpha_f = 2$. Solving the Compton equation for α_i,

$$\alpha_i = \frac{1}{\frac{1}{\alpha_f} - \left(1-\cos\theta\right)} \tag{6}$$

Physically, α_i is an energy and it must be positive – that is the most basic requirement we can make. In the equation above, the numerator is clearly always positive, so the only condition we can enforce is that the denominator remain positive. This requires

$$\frac{1}{\alpha_f} > (1 - \cos\theta) \tag{7}$$

If the denominator tends toward zero, α_i tends toward infinity, so this is equivalent to requiring that the incident photon have finite energy – also very sensible. Solving for θ,

$$\cos\theta > 1 - \frac{1}{\alpha_f} \tag{8}$$

$$\theta < \cos^{-1}\left(1 - \frac{1}{\alpha_f}\right) \tag{9}$$

In the last line, we reverse the inequality because $\cos\theta$ is a decreasing function of θ as θ increases from 0. Given that α_f must be at least two for pair production,

$$\theta < \cos^{-1}\left(1 - \frac{1}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \tag{10}$$

2. If we wish to observe an object which is 0.25 nm in size, what is the minimum-energy photon which can be used?

Solution: The resolution limit using photons will be - well within an order of magnitude anyway - the wavelength of the photons. If we need a resolution of 0.25 nm, we need a photon of this wavelength, or of energy

$$E = hf = \frac{hc}{\lambda} \approx \frac{1240\,\text{eV}\cdot\text{nm}}{0.25\,\text{nm}} \approx 4960\,\text{eV} = 4.96\,\text{keV} \tag{11}$$

By the way, it is handy to know that $hc \approx 1240\,\text{eV}\cdot\text{nm}$.

3. The resolving power of a microscope is proportional to the wavelength used. We desire a 10^{-11} m (0.01 nm) resolution in order to “see" an atom. If electrons are used, what minimum kinetic energy is required to reach this resolution? Do not assume that the electron can be treated without relativity.

Solution: The de Broglie relationship tells us $\lambda = h/p$. In order to resolve features of a certain size with a microscope, the probe we're using should have a wavelength at least the same size as the desired resolution (if not smaller, ideally). In this case, our probe is an electron beam, so we need to have an electron wavelength of at least 10^{-11} m to resolve features of that size. Using the de Broglie relationship, and assuming we may need to consider relativistic effects, we could write

$$\lambda = 10^{-11}\,\text{m} = \frac{h}{p} = \frac{h}{\gamma m v} \tag{12}$$

Of course, we want the kinetic energy, rather than the momentum, so we should make use of the relativistic energy-momentum relationship:

$$K = \sqrt{p^2c^2 + m^2c^4} - mc^2 = (\gamma - 1)\,mc^2 \tag{13}$$

We now have two choices: solve for the speed using Eq. 12, and then calculate K, or solve the whole thing algebraically first to put K in terms of p. We choose the latter.

$$K + mc^2 = \sqrt{p^2c^2 + m^2c^4} \tag{14}$$

$$\left(K + mc^2\right)^2 - m^2c^4 = p^2c^2 \tag{15}$$

$$p = \frac{1}{c}\sqrt{(K + mc^2)^2 - m^2c^4} \tag{16}$$

$$p = \frac{1}{c}\sqrt{\left[(K + mc^2 + mc^2)\,(K + mc^2 - mc^2)\right]} = \frac{1}{c}\sqrt{K\,(K + 2mc^2)} \tag{17}$$

Note the factorization on the last line. Inserting that into Eq. 12, and solving for K:

$$\lambda = \frac{hc}{\sqrt{K\,(K + 2mc^2)}} \tag{18}$$

$$\left(\frac{hc}{\lambda}\right)^2 = K\left(K + 2mc^2\right) \tag{19}$$

$$0 = K^2 + \left(2mc^2\right)K - \left(\frac{hc}{\lambda}\right)^2 \tag{20}$$

$$K = -mc^2 \pm \sqrt{(mc^2)^2 + \left(\frac{hc}{\lambda}\right)^2} \tag{21}$$

Clearly, only the positive root is physical. Note that the positive root always gives a positive kinetic energy, so long as λ is not zero. Using the numbers given, and noting $mc^2 = 511\,\text{keV}$ and $hc = 1240\,\text{eV}\cdot\text{nm}$,

$$K = \sqrt{(mc^2)^2 + \left(\frac{hc}{\lambda}\right)^2} - mc^2 = \sqrt{(511\,\text{keV})^2 + \left(\frac{1240\,\text{eV}\cdot\text{nm}}{0.01\,\text{nm}}\right)^2} - 511\,\text{keV} \approx 14.8\,\text{keV} \tag{22}$$

This means one needs to accelerate the electron through a potential difference of about 15 kV. Note also

$$K = mc^2\left(\sqrt{1+\left(\frac{hf}{mc^2}\right)^2}-1\right) = mc^2\left(\sqrt{1+\left(\frac{\lambda_c}{\lambda}\right)^2}-1\right) \tag{23}$$

Here it is more apparent that the relative energy scale is the photon energy hf divided by the electron's rest energy mc^2, and that the relevant distance scale is the electron's wavelength relative to its Compton wavelength $\lambda_c = h/mc$. When the electron's wavelength is of the same order as or smaller than the Compton wavelength, or its energy is comparable to or larger than its rest energy, relativistic and quantum effects become important.

4. By doing a nuclear diffraction experiment, you measure the de Broglie wavelength of a proton to be 9.16 fm. **(a)** What is the speed of the proton? **(b)** Through what potential difference must it be accelerated to achieve that speed?

Solution: Let us assume relativity is needed, since a femtometer is a very small distance scale, implying a large energy. We start with deBroglie, and add relativity:

$$\lambda = \frac{h}{p} = \frac{h}{\gamma mv} = \frac{h\sqrt{1-v^2/c^2}}{mv} \tag{24}$$

Now we solve for v

$$\lambda^2 m^2 v^2 = h^2 - \frac{h^2 v^2}{c^2} \tag{25}$$

$$h^2 = v^2\left(\lambda^2 m^2 + \frac{h^2}{c^2}\right) \tag{26}$$

$$v = \frac{hc}{\sqrt{h^2+\lambda^2 m^2 c^2}} = \frac{c}{\sqrt{1+(\lambda mc/h)^2}} = \frac{c}{\sqrt{1+(\lambda/\lambda_c)^2}} \approx 0.143c \tag{27}$$

Again we see the relevant distance scale is the Compton wavelength for the proton, $\lambda_c = h/m_pc \approx 1.32 \times 10^{15}$ m. Given the speed, the kinetic energy is no big deal, particularly noting that $m_pc^2 \approx 938$ MeV:

$$K = (\gamma - 1)\,mc^2 \approx 9.72\,\text{MeV} \tag{28}$$

Given the proton's charge of $+e$, this means we need to move it through a potential difference of -9.72 MV to reach the desired de Broglie wavelength.

5. A hydrogen atom is moving at a speed of 125.0 m/s. It absorbs a photon of wavelength 97 nm that is moving in the opposite direction. By how much does the speed of the atom change as a result of absorbing the photon?

Solution: Just conservation of momentum. Initially, we have the hydrogen's and photon's momentum, after we have just the hydrogen. We don't need relativity, given the low velocity compared to c.

$$p_{hi} + p_{ph} = p_{hf} \tag{29}$$

Given $p_{ph} = h/\lambda$ and $p_h = mv$, and noting the directions,

$$mv_i - \frac{h}{\lambda} = mv_f \tag{30}$$

$$v_f = v_i - \frac{h}{\lambda m} \tag{31}$$

$$\Delta v = v_f - v_i = \frac{h}{\lambda m} \tag{32}$$

Given a the mass of a hydrogen atom is about 1.67×10^{-27} m, $\Delta v \approx 4.1$ m/s.

6. Suppose an atom of iron at rest emits an X-ray photon of energy 6.4 keV. Calculate the "recoil" momentum and kinetic energy of the atom *Hint: do you expect to need classical or relativistic kinetic energy for the atom? Is the kinetic energy likely to be much smaller than the atom's rest energy?*

Solution: Same as the last problem. Do we need relativity? Only if the photon energy is comparable to the iron atom's rest energy. The latter has a mass of about 9.27×10^{-26} kg, implying a rest energy of $mc^2 \approx 52$ GeV. That's about ten billion times the photon energy, so we are probably good using classical physics. Now it is just conservation of momentum like the last problem: we start with an iron atom at rest ($p = 0$), end up with an iron atom and photon going in the opposite direction. The iron atom and photon must therefore have equal and opposite momentum.

$$p_{Fe} = p_{ph} \tag{33}$$

$$mv = \frac{h}{\lambda} = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}E_{ph} \approx 3.4 \times 10^{-24}\,\text{kg}\cdot\text{m/s} \tag{34}$$

This implies a recoil velocity of about 37 m/s. (Note that the thermal velocity at room temperature for an iron atom more like 370 m/s.) The recoil kinetic energy of the iron atom is then

$$K = \frac{p^2}{2m} \approx 3.9 \times 10^{-4}\,\text{eV} = 0.39\,\text{meV} \tag{35}$$

7. Consider a particle described by the wave function

$$\psi(x) = \frac{N}{x^2 + a^2}$$

(a) What is the probability $P(x)\,dx$ of finding the particle in the interval $[x, x + dx]$?
(b) We require that $\int_{\infty}^{\infty} P(x)\,dx = 1$. What value of N is required for this to be true?
(c) What is the expected value of the particle's position $\langle x \rangle$

Solution: The probability that the particle is in the interval $[x, x + dx]$ is just the square of the wave function in 1D:

$$P(x) = |\psi|^2 = \frac{N^2}{\left(x^2 + a^2\right)^2} \tag{36}$$

Normalization will find the value of N:

$$1 = \int_{-\infty}^{\infty} \frac{N^2}{\left(x^2 + a^2\right)^2}\,dx = \frac{N^2}{2a^2}\left(\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{a^2 + x^2}\right)\Bigg|_{-\infty}^{\infty} \tag{37}$$

$$1 = \frac{N^2}{2a^2}\left(\frac{\pi}{2a} - \frac{-\pi}{2a} + 0 - 0\right) = \frac{N^2\pi}{2a^3} \tag{38}$$

$$\implies \qquad N = \sqrt{\frac{2a^3}{\pi}} \tag{39}$$

With that out of the way, the expected position is easily calculated. By inspection, we can see that the integrand is an odd function in x, and will integrate to zero.

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi|^2 dx = \int_{-\infty}^{\infty} \frac{xN^2}{\left(x^2 + a^2\right)^2} dx = 0 \tag{40}$$

8. Consider the wave function from the preceding problem. **(a)** Find $\langle x^2 \rangle$ and Δx. **(b)** What is the probability the particle is in the interval $[-a, a]$?

Solution: The expected value of x^2 is straightforward enough; we will presume you have a table of integrals or Wolfram Alpha handy. We will need the value of N from the last problem.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2|\psi|^2\,dx = \int_{-\infty}^{\infty} \frac{x^2N^2}{\left(x^2 + a^2\right)^2}\,dx = N^2\frac{\pi}{2a} = \frac{2a^3}{\pi}\frac{\pi}{2a} = a^2 \tag{41}$$

With that in hand,

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{a^2 - 0} = a \tag{42}$$

We can therefore interpret the wave function from the previous problem as one which localizes a particle within a characteristic distance a of the origin. Of course, you may have recognized the original function (`http://en.wikipedia.org/wiki/Cauchy_distribution`), in which case this makes sense.

The probability the particle is in a region $[-a, a]$ can be found by integrating P from $-a$ to a.

$$P(x\in[-a,a]) = \int_{-a}^{a} |\psi|^2\,dx = \int_{-a}^{a} \frac{2a^3}{\pi}\frac{1}{\left(x^2+a^2\right)^2}\,dx = \frac{2a^3}{\pi}\frac{1}{2a^3}\left(\tan^{-1}\left(\frac{x}{a}\right) + \frac{ax}{a^2+x^2}\right)\Bigg|_{-a}^{a} \tag{43}$$

$$= \frac{1}{\pi}\left(\tan^{-1}(1) - \tan^{-1}(-1) + \frac{a^2}{a^2+a^2} - \frac{-a^2}{a^2+a^2}\right) = \frac{1}{\pi}\left(\frac{\pi}{2}+1\right) \tag{44}$$

$$= \frac{1}{2} + \frac{1}{\pi} \approx 0.818 \tag{45}$$

9. Consider an electron confined to a 1-dimensional box with infinitely high walls. We know that the allowed energies are discrete. However, in order to observe these discrete levels in an experiment, we should expect that their spacing must be large compared to the electron's thermal energy. Presuming we want to be able to resolve the difference between the first two energy levels, and a thermal energy of $\frac{1}{2}k_bT$, estimate the size of the largest "box" that our electron can be confined to at temperatures of **(a)** 295 K (room temperature), and **(b)** 4.2 K (boiling point of liquid helium). **(c)** How cold would it have to be to make the box 1 mm wide?

Solution: The energy levels of a particle confined to a 1D box are

$$E_n = \frac{n^2h^2}{8mL^2} \qquad n \in \{1, 2, 3...\} \tag{46}$$

The energy difference between the first two levels ($n=1$ and $n=2$) is then

$$\Delta E_{12} = E_2 - E_1 = \frac{3h^2}{8mL^2} \tag{47}$$

In order to resolve this difference in our experiment, we want the thermal energy to be smaller than this level spacing. The thermal energy of $\frac{1}{2}k_bT$ is random, meaning it is essentially an uncertainty on top of any measurement we make. If this thermal energy is larger than the level spacing, it means the uncertainty in our measurement is larger than the difference we're looking for, and we'll have no luck. At a given T, the thermal uncertainty is fixed, so all we can do is make the box smaller to make the levels easier to resolve. Thus, what we want is

$$\Delta E_{12} \geq \frac{1}{2} k_b T \tag{48}$$

$$\frac{3h^2}{8mL^2} \geq \frac{1}{2} k_b T \tag{49}$$

$$L \leq \sqrt{\frac{3h^2}{2mk_bT}} \tag{50}$$

At $T = 295\,\mathrm{K}$, we get a box size of approximately $L \lesssim 13\,\mathrm{nm}$, and at $4.2\,\mathrm{K}$ we find $L \lesssim 110\,\mathrm{nm}$. Both are experimentally achievable these days, though the latter is much, much easier (even indexed against the lower temperature requirement, $4.2\,\mathrm{K}$ is not so difficult).

On the other hand, if we want to have a $1\,\mathrm{mm}$ box, how cold does our system have to be? Just solve the equation above for T, with $L = 10^{-3}\,\mathrm{m}$.

$$T < \frac{3h^2}{2mL^2 k_b} \sim 52\,\mathrm{nK} \tag{51}$$

In fact, this isn't a totally impossible experiment (optical techniques allow cooling to nK temperatures), but clearly it is the case that making anything *macroscopic* exhibit quantum behavior is going to be tricky at best, and not something you'll see every day.

10. At time $t = 0$ a particle is represented by the wave function

$$\psi(x,0) = \begin{cases} Ax/a & 0 \leq x \leq a \\ a\,(b-x)\,/\,(b-a) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{52}$$

where A, a, and b are constants. **(a)** Normalize ψ (that is, find A in terms of a and b). **(b)** Sketch $\psi(x,0)$ as a function of x. **(c)** Where is the particle most likely to be found at $t = 0$? **(d)** What is the probability of finding the particle to the left of a? You can check your result in the limiting cases $b = a$ and $b = 2a$. **(e)** What is the expectation value of x?

Solution: Normalizing means integrating $|\psi|^2$ over all space, but since ψ is zero except over $[0, b]$, we need only perform the integral over that interval. We will assume A may be complex, but will presume a and b to be real.

$$1 = \frac{|A|^2}{a^2} \int_0^a x^2\, dx + \frac{|A|^2}{(b-a)^2} \int_a^b (b-x)^2 \ dx \tag{53}$$

$$= |A|^2 \left[\frac{1}{a^2} \left(\frac{x^3}{3} \right) \Big|_0^a + \frac{1}{(b-a)^2} \left(-\frac{(b-x)^3}{3} \right) \Big|_a^b \right] \tag{54}$$

$$= |A|^2 \left[\frac{a}{3} + \frac{b-a}{3} \right] = |A|^2 \frac{b}{3} \tag{55}$$

$$\implies \quad A = \sqrt{\frac{3}{b}} \tag{56}$$

In the end, A is purely real, but best to be on the safe side. The wave function is linearly increasing from 0 to a, going from $\psi(0) = 0$ to $\psi(a) = A$, and linearly decreasing from a to b, going from $\psi(a) = A$ to $\psi(b) = 0$. A sketch is left as an exercise to the reader.

The particle is most likely to be found where $|\psi|^2$ has a maximum, which in this case means the place where ψ itself is maximum. From the given form (or your sketch), the maximum is clearly at $x = a$.

The probability of finding the particle to the left of a is given by integrating the probability density from $-\infty$ to a:

$$P(\text{in } [-\infty, a]) = \frac{|A|^2}{a^2} \int_0^a x^2\, dx = \frac{3}{ba^2} \frac{x^3}{3} \Big|_0^a = \frac{a}{b} \tag{57}$$

In the limiting case that $b = a$, $P = 1$. In this case the wavefunction is zero for $x > a$, so the particle has no where else to be. In the limiting case $b = 2a$, the wavefunction is defined over two symmetric intervals, with $x = a$ right in the middle. By symmetry, the particle must be to the left of a half of the time, and we indeed find $P = \frac{1}{2}$.

The expectation value is found by integrating $x|\psi|^2$ over all space, or over $[0, b]$ since ψ is zero elsewhere.

$$\langle x \rangle = \int x|\psi|^2\,dx = \frac{|A|^2}{a^2}\int\limits_0^a x^3\,dx + \frac{|A|^2}{(b-a)^2}\int\limits_a^b x\,(b-x)^2\;dx \tag{58}$$

$$= \frac{3}{b}\left[\frac{1}{a^2}\left(\frac{x^4}{4}\right)\Big|_0^a + \frac{1}{(b-a)^2}\left(b^2\frac{x^2}{2} - 2b\frac{x^3}{3} + \frac{x^4}{4}\right)\Big|_a^b\right] \tag{59}$$

$$= \frac{3}{4b\,(b-a)^2}\left[a^2\,(b-a)^2 + 2b^4 - \frac{8b^4}{3} - 2a^2b^2 + \frac{8a^3b}{3} - a^4\right] \tag{60}$$

$$= \frac{3}{4b\,(b-a)^2}\left(\frac{b^3}{3} - a^2b^2 + \frac{2}{3}a^3b\right) = \frac{1}{4\,(b-a)^2}\left(b^3 - 3a^2b + 2a^3\right) = \frac{2a+b}{4} \tag{61}$$

Wolfram Alpha will tell you that $b^3 - 3a^2b + 2a^3$ factors to $(a-b)^2\,(2a+b)$ if you didn't notice. The last step is only valid if $a \neq b$, which seems sensible based on the definition of the wavefunction given.

11. Nuclei, typically of size 10^{-14} m, frequently emit electrons, with typical energies of $1-10$ MeV. Use the uncertainty principle to show that electrons of energy 1 MeV could not be contained in the nucleus before the decay.

Solution: Clearly we need to use the uncertainty principle. We are given a characteristic region in which the electron is confined, which we can interpret as the minimum uncertainty in position, $\Delta x = 10^{-14}$ m. The uncertainty principle will get us the corresponding Δp, which we can relate to the electron's total energy. We're obliged to use the relativistic energy-momentum relationship, since the energy in question is far above the electron's rest energy ($m_e c^2 \approx 511$ keV).

One more question: should we use total energy or kinetic energy? For the purposes of this problem and our general conclusion, it doesn't make much difference, but let us assume that what experimenters are able to measure is the kinetic energy. This amounts to saying that the rest energy is unchanged before and after decay, so it would cancel.

First, we can find the minimum uncertainty in momentum:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \qquad \Longrightarrow \qquad \Delta p \geq \frac{\hbar}{2\Delta x} \tag{62}$$

If Δp is the minimum *uncertainty* in momentum, then this is also the minimum value for momentum: $p_{\text{min}} = \Delta p$. Noting that, and using the relativistic kinetic energy-momentum relationship (total energy minus rest energy),

$$KE_{\text{min}} = \sqrt{m^2c^4 + p_{\text{min}}^2c^2} - mc^2 = \sqrt{m^2c^4 + \frac{\hbar^2c^2}{4\,(\Delta x)^2}} - mc^2 \approx 62\,\text{MeV} \tag{63}$$

An electron would require a kinetic energy of at least 62 MeV to be confined to a region of 10^{-14} m, far more than the 1–10 MeV the observed electrons have. This is a big hint as to how nuclear decay has to work ...

Incidentally, knowing that the electron's energy is well above its rest energy, we could neglect the m^2c^4 and mc^2 rest energy terms, and simplify things a bit. This amounts to saying $K \approx pc$ for highly relativistic electrons.

$$KE_{\text{min}} \approx p_{\text{min}}^2 c^2 = \frac{\hbar c}{2\Delta x} \approx 62\,\text{MeV} \tag{64}$$

We can also pose the question another way: what would the confinement size Δx have to be to be consistent with the observed electron energies? We just have to solve for Δx instead.

$$\Delta x = \frac{\hbar c}{\sqrt{4\left[(KE_{\text{min}} + mc^2)^2 - m^2c^4\right]}} \tag{65}$$

If we know $KE_{\text{min}} \gg mc^2$, we can simplify this to $\Delta x \approx \hbar c/2KE_{\text{min}}$ to very good accuracy. For 10 MeV electrons, we find $\Delta x \approx 6 \times 10^{-14}$ m, nearly an order of magnitude larger (and we know nuclei aren't this large). For 1 MeV electrons, we find $\Delta x \approx 4 \times 10^{-13}$ m, nearly two orders of magnitude larger.

An interesting read related to this problem, and fundamental length scales in general can be found here: `http://math.ucr.edu/home/baez/lengths.html`.

12. From the relationship $\Delta p \Delta x \geq h/4\pi$, show that for a particle moving in a circle $\Delta L\, \Delta\theta \geq h/4\pi$. The quantity ΔL is the uncertainty in angular momentum and $\Delta\theta$ is the uncertainty in the angle.

Solution: An electron traveling in a circular path of radius r covering a distance Δx along the circle's perimeter moves through an angle $\Delta\theta$ according to the arclength formula $\Delta x = \Delta\theta r$. Since r is fixed, this formula holds equally well for an uncertainty in position. The momentum of a the particle is $p = mv = mr\omega$, and given that m and r are constants, any uncertainty Δp in the particle's momentum must come from an uncertainty in ω, $\Delta p = mr\Delta\omega$.

On the other hand, the particle's angular momentum is $L = mvr = mr^2\omega$, and its uncertainty can only come from an uncertainty in ω, $\Delta L = mr^2\Delta\omega$. Putting our relationships together,

$$\Delta x\, \Delta p = (r\Delta\theta)\,(mr\Delta\omega) = \Delta\theta \left(mr^2\Delta\omega\right) = \Delta\theta\, \Delta L \geq \frac{h}{4\pi} \tag{66}$$

13. Given the wave function

$$\psi(x) = \begin{cases} Ne^{\kappa x} & x < 0 \\ Ne^{-\kappa x} & x > 0 \end{cases} \tag{67}$$

(a) Find N needed to normalize ψ.
(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and Δx.

Solution: In order to normalize the wavefunction, we need to split up the usual integral into two integrals over $[-\infty, 0]$ and $[0, \infty]$ since the function is defined separately over those intervals. Since the wave function is piecewise continuous, this need not trouble us though.

$$1 = \int_{-\infty}^{\infty} |\psi|^2 \, dx = \int_{-\infty}^{0} |\psi|^2 \, dx + \int_{-0}^{\infty} |\psi|^2 \, dx = \int_{-\infty}^{0} N^2 e^{2\kappa x} \, dx + \int_{-0}^{\infty} N^2 e^{-2\kappa x} \, dx \tag{68}$$

$$= N^2 \left[\frac{1}{2\kappa} e^{2\kappa x} \Big|_{-\infty}^{0} + \frac{1}{2\kappa} e^{-2\kappa x} \Big|_{0}^{\infty} \right] = \frac{N^2}{\kappa} \tag{69}$$

$$\implies \qquad N = \sqrt{\kappa} \tag{70}$$

Next, we find $\langle x \rangle$ in the usual way, again taking care to split the integral into two bits:

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi|^2 \, dx = \int_{-\infty}^{0} xN^2 e^{2\kappa x} \, dx + \int_{-0}^{\infty} xN^2 e^{-2\kappa x} \, dx = 0 \tag{71}$$

By symmetry, the two integrals are equal in magnitude and opposite in sign, so the expected position is at the origin. Finding $\langle x^2 \rangle$ requires only a bit more math:

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2|\psi|^2 \, dx = \int_{-\infty}^{0} x^2 N^2 e^{2\kappa x} \, dx + \int_{-0}^{\infty} x^2 N^2 e^{-2\kappa x} \, dx \tag{72}$$

$$= \frac{N^2}{4\kappa^3} \left(2\kappa^2 x^2 - 2\kappa x + 1 \right) e^{2\kappa x} \Big|_{-\infty}^{0} + \frac{N^2}{4\kappa^3} \left(2\kappa^2 x^2 + 2\kappa x + 1 \right) \left(-e^{-2\kappa x} \right) \Big|_{0}^{\infty} \tag{73}$$

$$= \frac{N^2}{2\kappa^3} = \frac{1}{2\kappa^2} \tag{74}$$

Given $\langle x \rangle$ and $\langle x^2 \rangle$,

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2}\kappa} \tag{75}$$

14. An electron in a hydrogen atom is in a state described by the wave function

$$\psi = \frac{1}{\sqrt{3}\,(2a_o)^{3/2}}\frac{r}{a_o}e^{-r/2a_o} \tag{76}$$

where a_o is the Bohr radius.

(a) What is the *most probable* value of r?
(b) What is $\langle r\rangle$?

Solution: The most likely distance corresponds to the distance at which the probability of finding the electron is maximum. This is distinct from the expected value of the radius $\langle r\rangle$. The probability of finding an electron at a distance r in the interval $[r, r+dr]$, in spherical coordinates, is the squared magnitude of the wavefunction times the volume of a spherical shell of thickness dr and radius r:

$$P(r)\,dr = |\psi|^2 \cdot 4\pi r^2\,dr \qquad \text{or} \qquad P(r) = |\psi|^2 \cdot 4\pi r^2 \tag{77}$$

Given ψ above, we have

$$P(r) = \left|\frac{1}{\sqrt{3}\,(2a_o)^{3/2}}\frac{r}{a_o}e^{-r/2a_o}\right|^2 \cdot 4\pi r^2 = \frac{\pi r^4}{6a_o^5}e^{-r/a_o} \tag{78}$$

The most probable radius is when $P(r)$ takes a maximum value, which must occur when $dP/dr=0$ and $d^2P/dr^2<0$. Thus:

$$\frac{dP}{dr} = 0 = \left(\frac{\pi}{6a_o^5}\right)\frac{d}{dr}\left(r^4e^{-r/a_o}\right) = \left(\frac{\pi}{6a_o^5}\right)\left(4r^3e^{-r/a_o} - \frac{r^4}{a_o}e^{-r/a_o}\right) \tag{79}$$

$$0 = \left(\frac{\pi r^3}{6a_o^5}e^{-r/a_o}\right)\left(4 - \frac{r}{a_o}\right) \tag{80}$$

$$\Longrightarrow \qquad r = \{0, 4a_o, \infty\} \tag{81}$$

One can either apply the second derivative test or make a quick plot of $P(r)$ to verify that $r=4a_o$ is the sole maximum of the probability distribution, and hence the most probable radius, while $r=0$ and $r=\infty$ are minima.

On to $\langle r\rangle$. We must first verify that the wave function is normalized to get a correct value for $\langle r\rangle$ – this did not matter for the most likely value of r, since we differentiated the wave function and any overall normalization constants are irrelevant. Let us now define an overall constant multiplier for the wave function A to fix normalization and find its value. That is, let $\psi \to A\psi$ and enforce normalization to find A.

$$1 = \int_0^\infty A^2 P(r)\,dr = A^2 \int_0^\infty \frac{\pi r^4}{6a_o^5} e^{-r/a_o}\,dr = \frac{A^2\pi}{6a_o} \int_0^\infty \frac{r^4}{a_o^4} e^{-r/a_o}\,dr \tag{82}$$

At this point, it most clever to change variables to $u = r/a_o$, so $du = dr/a_o$, which gives us a well-known integral:

$$1 = \frac{A^2\pi}{6a_o} \int_0^\infty \frac{r^4}{a_o^4} e^{-r/a_o}\,dr = \frac{A^2\pi}{6} \int_0^\infty u^4 e^{-u}\,du = \frac{A^2\pi}{6}\frac{4!}{1^5} = 4\pi A^2 \tag{83}$$

$$\implies \qquad A^2 = \frac{1}{4\pi} \tag{84}$$

Now we can find $\langle r \rangle$ correctly. Since the wave function is spherically symmetric, we can just integrate over radius using the volume element $dV = 4\pi r^2\,dr$:

$$\langle r \rangle = \int_0^\infty r A^2 |\psi|^2\, 4\pi r^2\,dr = \int_0^\infty \frac{r^5}{24a_o^5} e^{-r/a_o}\,dr = \frac{1}{24} \int_0^\infty \left(\frac{r}{a_o}\right)^5 e^{-r/a_o}\,dr \tag{85}$$

Again, at this point, it most clever to change variables to $u = r/a_o$, so $du = dr/a_o$, which again gives us a well-known integral:

$$\langle r \rangle = \frac{a_o}{24} \int_0^\infty u^5 e^{-u}\,du = \frac{a_o}{24}\frac{5!}{1^6} = 5a_o \tag{86}$$

Of course, we did not check that this wave function is normalized.

Constants:

$$N_A = 6.022 \times 10^{23} \text{ things/mol}$$

$$k_e \equiv 1/4\pi\epsilon_o = 8.98755 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2}$$

$$\epsilon_o = 8.85 \times 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2$$

$$\mu_0 \equiv 4\pi \times 10^{-7} \text{ T} \cdot \text{m/A}$$

$$e = 1.60218 \times 10^{-19} \text{ C}$$

$$h = 6.6261 \times 10^{-34} \text{ J} \cdot \text{s} = 4.1357 \times 10^{-15} \text{ eV} \cdot \text{s}$$

$$\hbar = \frac{h}{2\pi} \qquad hc = 1239.84 \text{ eV} \cdot \text{nm}$$

$$k_B = 1.38065 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} = 8.6173 \times 10^{-5} \text{ eV} \cdot \text{K}^{-1}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.99792 \times 10^8 \text{ m/s}$$

$$m_e = 9.10938 \times 10^{-31} \text{ kg} \qquad m_e c^2 = 510.998 \text{ keV}$$

$$m_p = 1.67262 \times 10^{-27} \text{ kg} \qquad m_p c^2 = 938.272 \text{ MeV}$$

$$m_n = 1.67493 \times 10^{-27} \text{ kg} \qquad m_n c^2 = 939.565 \text{ MeV}$$

Schrödinger

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + V(x)\Psi \quad \text{1D time-dep}$$

$$E\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi \quad \text{1D time-indep}$$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1 \qquad P(\text{in } [x, x+dx]) = |\psi(x)|^2 \qquad \text{1D}$$

$$\int_0^{\infty} |\psi(r)|^2 \, 4\pi r^2 \, dr = 1 \qquad P(\text{in } [r, r+dr]) = 4\pi r^2 |\psi(r)|^2 \qquad \text{3D}$$

$$\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P(x) \, dx \qquad \text{1D} \qquad \langle r^n \rangle = \int_0^{\infty} r^n P(r) \, dr \qquad \text{3D}$$

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

Basic Equations:

$$\vec{\mathbf{F}}_{\text{net}} = m\vec{\mathbf{a}} \quad \text{Newton's Second Law}$$

$$\vec{\mathbf{F}}_{\text{centr}} = -\frac{mv^2}{r}\hat{\mathbf{r}} \quad \text{Centripetal}$$

$$\vec{\mathbf{F}}_{12} = k_e \frac{q_1 q_2}{r_{12}^2} \hat{\mathbf{r}}_{12} = q_2 \vec{\mathbf{E}}_1 \qquad \vec{\mathbf{r}}_{12} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$$

$$\vec{\mathbf{E}}_1 = \vec{\mathbf{F}}_{12}/q_2 = k_e \frac{q_1}{r_{12}^2} \hat{\mathbf{r}}_{12}$$

$$\vec{\mathbf{F}}_B = q\vec{\mathbf{v}} \times \vec{\mathbf{B}}$$

$$0 = ax^2 + bx^2 + c \Longrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Approximations, $x \ll 1$

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n+1)x^2 \qquad \tan x \approx x + \frac{1}{3}x^3$$

$$e^x \approx 1 + x + \frac{1}{2}x \qquad \sin x \approx x - \frac{1}{6}x^3 \qquad \cos x \approx 1 - \frac{1}{2}x^2$$

Misc Quantum

$$E = hf \qquad p = h/\lambda = E/c \qquad \lambda f = c \qquad \text{photons}$$

$$\lambda_f - \lambda_i = \frac{h}{m_e c}(1 - \cos\theta)$$

$$\lambda = \frac{h}{|\vec{\mathbf{p}}|} = \frac{h}{\gamma m v} \approx \frac{h}{mv}$$

$$\Delta x \Delta p \geq \frac{h}{4\pi} \qquad \Delta E \Delta t \geq \frac{h}{4\pi}$$

$$eV_{\text{stopping}} = KE_{\text{electron}} = hf - \varphi = hf - W$$

Bohr

$$\begin{aligned}
E_n &= -13.6\,\text{eV}/n^2 \qquad \text{Hydrogen} \\
E_n &= -13.6\,\text{eV}\left(Z^2/n^2\right) \qquad Z \text{ protons, 1 e}^- \\
E_i - E_f &= -13.6\,\text{eV}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = hf \\
L = mvr &= n\hbar \\
v^2 &= \frac{n^2\hbar^2}{m_e^2 r^2} = \frac{k_e e^2}{m_e r}
\end{aligned}$$

Calculus of possible utility:

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int u dv = uv - \int v du$$

$$\int \sin ax dx = -\frac{1}{a}\cos ax + C$$

$$\int \cos ax dx = \frac{1}{a}\sin ax + C$$

$$\frac{d}{dx}\tan x = \sec^2 x = \frac{1}{\cos^2 x}$$

$$\int e^{-ax} \, dx = -\frac{1}{a}e^{-ax} + C$$

$$\int_0^{\infty} x^n e^{-ax} \, dx = \frac{n!}{a^{n+1}}$$

$$\int_0^{\infty} x^2 e^{-ax^2} \, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$

$$\int_{-\infty}^{\infty} x^3 e^{-ax^2} \, dx = \int_{-\infty}^{\infty} x e^{-ax^2} \, dx = 0$$

$$\int_0^{\infty} x^4 e^{-ax^2} \, dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$