University of Alabama
Department of Physics and Astronomy

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Exam 1 practice problems: answers & some solutions

1. Show that the angular frequency of a charge moving in a uniform magnetic field is given by

$$\omega = \frac{qB}{m}\sqrt{1-\frac{u^2}{c^2}} \tag{1}$$

In problem 1 on homework 2, we derived the condition for relativistic circular motion

$$r = \frac{\gamma m v}{qB} \qquad \text{or} \qquad v = \frac{qBr}{\gamma m} \tag{2}$$

Noting that for uniform circular motion $v\!=\!r\omega$,

$$r\omega = \frac{qBr}{\gamma m} \qquad \Longrightarrow \qquad \omega = \frac{qB}{\gamma m} = \frac{qB}{m}\sqrt{1-\frac{v^2}{c^2}} \tag{3}$$

2. An excited atom of mass m, initially at rest in frame S, emits a photon and recoils. The internal energy of the atom decreases by ΔE and the energy of the photon is hf. Show that

$$hf = \Delta E\left(1-\frac{\Delta E}{2mc^2}\right) \tag{4}$$

For this problem, we need to make use of the fact that a change in internal energy is equivalent to a change in mass. From mass-energy equivalence, the photon carrying away energy from the excited atom is equivalent to the photon carrying away mass, so the recoiling atom should have a slightly smaller mass than it started with. If the starting mass is m and the ending mass m', conservation of energy gives

$$E = mc^2 = m'c^2 + hf = E' + hf \tag{5}$$

Since the atom is initially at rest, it has only its rest energy $E = mc^2$. After photon emission, atom's mass-energy E' is smaller by an amount equal to the photon's energy. With the atom initially at rest, momentum conservation is straightforward

$$0 = m'v - \frac{hf}{c} = p' - \frac{hf}{c} \tag{6}$$

For the recoiling atom, we can make use of the total energy expression $E'^2 = p'^2c^2 + m'^2c^4$ and substitute our energy equation $E' = mc^2 - hf$:

$$\begin{aligned} \left(m'c^2\right)^2 &= E'^2 - \left(cp'\right)^2 && (7) \\ &= \left(mc^2 - hf\right)^2 - (hf)^2 && (8) \\ &= \left(mc^2\right)^2 - 2mc^2hf && (9) \end{aligned}$$

The change in energy ΔE is just the difference between initial and final energies for the atom, $\Delta E = E - E' = mc^2 - mc'^2$, which gives us after solving for $m'c^2$ and squaring,

$$\left(m'c^2\right)^2 = \left(mc^2\right)^2 - 2mc^2\Delta E + (\Delta E)^2 \tag{10}$$

Comparing the last two equations,

$$\begin{aligned} -2mc^2hf &= -2mc^2\Delta E + (\Delta E)^2 && (11) \\ hf &= \Delta E - \frac{\left(\Delta E^2\right)}{2mc^2} = \Delta E\left(1 - \frac{\Delta E}{2mc^2}\right) && (12) \end{aligned}$$

3. An electron moving initially with constant speed v is brought to rest with uniform deceleration a lasting for a time $t = v/a$. Compare the electromagnetic energy radiated during the deceleration with the electron's initial kinetic energy. Express the ratio in terms of two lengths, the distance light travels in time t and the classical electron radius, $r_e = \frac{e^2}{4\pi\epsilon_o m_e c^2} = \frac{k_e e^2}{m_e c^2}$. Recall the Larmor formula, which gives the total power radiated from an accelerating charge ($v \ll c$):

$$P = \frac{e^2a^2}{6\pi\epsilon_o c^3} = \frac{2k_e e^2 a^2}{3c^3} \tag{13}$$

The acceleration is $a = v/t$, so we can find the radiated power readily. After the time t required to decelerate completely, the radiated energy is $\Delta E = Pt$ and the initial kinetic energy of the electron is $\frac{1}{2}mv^2$. Thus,

$$\frac{\Delta E}{K} = \frac{2k_e e^2 a^2 t}{3c^3}\frac{1}{\frac{1}{2}mv^2} = \frac{4k_e e^2}{3c^3 mt} \tag{14}$$

The time light travels in time t is $r_l = ct$. Using the definition of the classical electron radius above,

$$\frac{\Delta E}{K} = \frac{k_e e^2}{mc^2}\left(\frac{4}{3ct}\right) = \frac{4r_e}{3r_l} \tag{15}$$

4. An atomic clock aboard a spaceship runs slow compared to an Earth-based atomic clock at a rate of 1.0 second per day. What is the speed of the spaceship?

The proper time t_p is that measured on earth, while a dilated time $t = \gamma t_p$ is measured on the ship. If the clock aboard the ship is 1 s per day slow, then using $1\,\text{day} = 86400\,\text{s}$

$$\frac{t - t_p}{t_p} = \frac{\gamma t_p - t_p}{t_p} = \gamma - 1 = \frac{1\,\text{s}}{86400\,\text{s}} \tag{16}$$

This gives $\gamma \approx 1 + 1.16 \times 10^{-5}$, or $v/c \approx 4.8 \times 10^{-3}$.

5. The speed of light with respect to a medium is $v'_x = c/n$, where n is the index of refraction. Suppose that the medium, say, flowing water, is moving past a stationary observer in the same direction as the light with speed V. Show that the observer measures the speed of light to be approximately

$$v_x = \frac{c}{n} + \left(1 - \frac{1}{n^2}\right)V \tag{17}$$

This effect was first observed by Fizeau in 1851.

This is just velocity addition in disguise. The velocity in the lab v_x is the relativistic sum of the velocity of the of the flowing water and that of light:

$$v_x = \frac{V + v'_x}{1 + Vv'_x/c^2} = \frac{v + c/n}{1 + v/cn} \tag{18}$$

If we restrict ourselves to $v/c \ll 1$, then we may approximate $1/(1 + v/cn) \approx 1 - v/nc$:

$$v_x = \frac{v + c/n}{1 + v/cn} \approx \left(v + \frac{c}{n}\right)\left(1 - \frac{v}{cn}\right) = v - \frac{v^2}{cn} + \frac{c}{n} - \frac{v}{n^2} \tag{19}$$

Since we have already required $v/c \ll 1$, we may neglect terms of order v^2, and we have

$$v_x \approx \frac{c}{n} + v\left(1 - \frac{1}{n^2}\right) \tag{20}$$

6. A mass M at rest decays into two particles of mass m_1 and m_2. Show that the magnitude of the momentum of each of the two particles is

$$\frac{p}{c} = \frac{\sqrt{M^2 - (m_1 + m_2)^2}\sqrt{M^2 - (m_1 - m_2)^2}}{2M} \tag{21}$$

Hint: conserve energy and momentum, and use $E^2 = m^2c^4 + p^2c^2$.

A lot of algebra in this one. First, conservation of momentum implies that the decay products m_1 and m_2 will have equal and opposite momentum, which we will just call p. Conservation of energy relates the rest mass of the initial particle Mc^2 to the total energy of the two decay products:

$$\begin{aligned} Mc^2 &= \sqrt{p^2c^2 + m_1^2c^4} + \sqrt{p^2c^2 + m_2^2c^4} \\ M &= \sqrt{p^2/c^2 + m_1^2} + \sqrt{p^2/c^2 + m_2^2} \end{aligned} \tag{22}$$

Square both sides, rearrange and square again. Much algebra and factoring ensues.

$$\begin{aligned}
&\frac{2p^2}{c^2} + m_1^2 + m_2^2 + 2\sqrt{\left(p^2/c^2 + m_1^2\right)\left(p^2/c^2 + m_2^2\right)} = M^2 \\
&4\left(\frac{p^2}{c^2} + m_1^2\right)\left(\frac{p^2}{c^2} + m_2^2\right) = \left[M^2 - \left(m_1^2 + m_2^2\right) - \frac{2p^2}{c^2}\right]^2 \\
&\frac{4p^4}{c^4} + 4\left(m_1^2 + m_2^2\right)\frac{p^2}{c^2} + 4m_1^2m_2^2 = \left[M^2 - \left(m_1^2 + m_2^2\right)\right]^2 - \frac{4p^2}{c^2}M^2 + \frac{4p^2}{c^2}\left(m_1^2 + m_2^2\right) + \frac{4p^4}{c^4} \\
&\frac{4p^2}{c^2}M^2 = \left[M^2 - \left(m_1^2 + m_2^2\right)\right]^2 - 4m_1^2m_2^2 \\
&\frac{4p^2}{c^2}M^2 = M^4 - 2M^2\left(m_1^2 + m_2^2\right) + \left(m_1^2 + m_2^2\right)^2 - 4m_1^2m_2^2 \\
&\frac{4p^2}{c^2}M^2 = M^4 - 2M^2\left(m_1^2 + m_2^2\right) + \left(m_1^2 - m_2^2\right)^2 \\
&\frac{4p^2}{c^2}M^2 = M^4 - 2M^2\left(m_1^2 + m_2^2\right) + (m_1 - m_2)^2(m_1 + m_2)^2 \\
&\frac{4p^2}{c^2}M^2 = M^4 - M^2\left[(m_1 - m_2)^2 + (m_1 + m_2)^2\right] + (m_1 - m_2)^2(m_1 + m_2)^2 \\
&\frac{p}{c} = \frac{\sqrt{M^2 - (m_1 + m_2)^2}\sqrt{M^2 - (m_1 - m_2)^2}}{2M}
\end{aligned} \tag{23}$$

7. For an experiment on the Compton effect, you want the X rays emerging at 90° from the incident direction to suffer an increase of wavelength by a factor 2. What wavelength to you need for your incident X rays?

Write the Compton equation in terms of the ratio $\lambda_i/\lambda_f = 1/2$ and substitute $\theta = 90°$:

$$\lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta) = \frac{h}{mc} = \lambda_c \tag{24}$$

$$\lambda_f \left(1 - \frac{\lambda_i}{\lambda_f}\right) = \lambda_c \tag{25}$$

$$\frac{1}{2}\lambda_f = \lambda_c \tag{26}$$

$$\implies \lambda_f = 2\lambda_c \tag{27}$$

$$\implies \lambda_i = \lambda_c \approx 2.43 \times 10^{-12}\,\mathrm{m} \tag{28}$$

8. X-ray photons of wavelength 0.154 nm are produced by a copper source. Suppose that 1.00×10^{18} of these photons are absorbed by the target each second.

(a) What is the total momentum p transferred to the target each second?
(b) What is the total energy E of the photons absorbed by the target each second?
(c) For these values, verify that the force on the target is related to the rate of energy transfer by

$$\frac{dp}{dt} = \frac{1}{c}\frac{dE}{dt} \tag{29}$$

The total momentum transfer per second is the number of absorbed photons per second (given) times the momentum per photon, h/λ:

$$\frac{dp}{dt} = \left(1.00 \times 10^{18}\,\text{photons/sec}\right)\frac{h}{\lambda} = \frac{10^{18}\left(6.63 \times 10^{-34}\right)}{0.154 \times 10^{-9}} \approx 4.31 \times 10^{-6}\,\mathrm{N} \tag{30}$$

The energy absorbed per second is the energy per photon times the number of photons absorbed per second:

$$\frac{dE}{dt} = \left(1.00 \times 10^{18}\,\text{photons/sec}\right)\frac{hc}{\lambda} = \frac{10^{18}\left(6.63 \times 10^{-34}\right)\left(3 \times 10^{8}\right)}{0.154 \times 10^{-9}} \approx 1.29 \times 10^{3}\,\mathrm{J/s} \tag{31}$$

From the equations above, we find $(dE/dt)\,/\,(dp/dt) = c$.

9. An interstellar space probe is moving at a constant speed relative to earth of $0.76c$ toward a distant planet. Its radioisotope generators have enough energy to keep its data transmitter active continuously for 15 years, as measured in their own reference frame. **(a)** How long do the generators last as measured from earth? **(b)** How far is the probe from earth when the generators fail, as measured from earth? **(c)** How far is the probe from earth when the generators fail, *as measured by its built-in trip odometer*?

Just to be clear, we will label quantities measured in the earth's reference frame with primes ($'$), and quantities without primes are with respect to the probe's reference frame. The relative velocity between the earth and the probe is the same from both reference frames, $v = v'$. From the probe's (and its generators') reference frame, it is the observers on earth that are moving. The observers on earth should then see a *longer* time interval compared to the proper time measured on the probe:

$$\Delta t' = \gamma \Delta_p = \frac{15\,\text{yrs}}{\sqrt{1 - \frac{(0.76c)^2}{c^2}}} \approx 23\,\text{yrs}$$

According to observers on earth, the generators should fail after a period of $\Delta t'$. Also according to them, the probe should have traveled a distance $d' = v'\Delta t'$ - the earth-bound observers watched the probe travel for an interval $\Delta t'$ at a constant velocity of v' in their reference frame:

$$d' = v'\Delta t' = (23\,\text{yrs})\left(3 \times 10^8\,\text{m/s}\right) \approx 2.2 \times 10^{17}\,\text{m}$$

Alternatively, we could express the distance in light years - the distance light travels in one year. To do that, we just have to realize that $0.76c$ means the probe travels at 76% of the speed of light:

$$d' = (0.76\,\text{light speed})\,(23\,\text{yrs}) \approx 18\,\text{light-years}$$

Finally, how about the distance traveled according to the probe? That is just the relative velocity multiplied by the elapsed time *from the probe's reference frame, i.e.,* the proper time:

$$d = v\Delta t = (15\,\text{yrs})\left(3 \times 10^8\,\text{m/s}\right)(0.76) = 1.1 \times 10^{17}\,\text{m} = 11\,\text{light-years}$$

10. A pion at rest ($m_\pi = 273\, m_{e^-}$) decays to a muon ($m_\mu = 207\, m_{e^-}$) and an antineutrino ($m_{\overline{\nu}} \approx 0$). This reaction is written as $\pi^- \rightarrow \mu^- + \overline{\nu}$. Find the kinetic energy of the muon and the energy of the antineutrino in electron volts. *Hint: relativistic momentum is conserved.*

Before the collision, we have only the pion, and since it is at rest, it has zero momentum and zero kinetic energy. After it decays, we have a muon and an antineutrino created and speed off in opposite directions (to conserve momentum). Both total energy - including rest energy - and momentum must be conserved before and after the collision.

First, conservation of momentum. Before the decay, since the pion is at rest, we have zero momentum. Therefore, afterward, the muon and antineutrino must have equal and opposite momenta. This means we can essentially treat this as a one-dimensional problem, and not bother with vectors. A consolation prize of sorts.

$$\text{initial momentum} = \text{final momentum} \tag{32}$$
$$p_\pi = p_\mu + p_\nu \tag{33}$$
$$0 = p_\mu + p_\nu \tag{34}$$
$$\implies p_\nu = -p_\mu = -\gamma_\mu m_\mu v_\mu \tag{35}$$

For the last step, we made use of the fact that relativistic momentum is $p = \gamma m v$. Now we can also write down conservation of energy. Before the decay, we have only the rest energy of the pion. Afterward, we have the energy of both the muon and antineutrino. The muon has both kinetic energy and rest energy, and we can write its total energy in terms of γ and its rest mass, $E = \gamma m c^2$. The antineutrino has negligible mass, and therefore no kinetic energy, but we can still assign it a total energy based on its momentum, $E = pc$.

$$\text{initial energy} = \text{final energy} \tag{36}$$
$$E_\pi = E_\mu + E_\nu \tag{37}$$
$$m_\pi c^2 = \gamma_\mu m_\mu c^2 + p_\nu c \tag{38}$$
$$m_\pi = \gamma_\mu m_\mu + \frac{p_\nu}{c} \tag{39}$$

Now we can combine these two conservation results and try to solve for the velocity of the muon:

$$m_\pi = \gamma_\mu m_\mu + \frac{p_\nu}{c} = \gamma_\mu m_\mu - \gamma_\mu m_\mu \frac{v_\mu}{c} \tag{40}$$
$$\frac{m_\pi}{m_\mu} = \gamma_\mu - \gamma_\mu \frac{v_\mu}{c} = \gamma \left[1 - \frac{v_\mu}{c}\right] \tag{41}$$

We will need to massage this quite a bit more to solve for v_μ ...

$$\frac{m_\pi}{m_\mu} = \gamma\left[1 - \frac{v_\mu}{c}\right] = \frac{1 - \frac{v_\mu}{c}}{\sqrt{1 - \frac{v_\mu^2}{c^2}}} \tag{42}$$

$$\left(\frac{m_\pi}{m_\mu}\right)^2 = \frac{\left(1 - \frac{v_\mu}{c}\right)^2}{1 - \frac{v_\mu^2}{c^2}} = \frac{\left(1 - \frac{v_\mu}{c}\right)^2}{\left(1 - \frac{v_\mu}{c}\right)\left(1 + \frac{v_\mu}{c}\right)} = \frac{1 - \frac{v_\mu}{c}}{1 + \frac{v_\mu}{c}} \tag{43}$$

Now we're getting somewhere. Take what we have left, and solve it for v_μ ... we will leave that as an exercise to the reader, and quote only the result, using the given masses of the pion and muon:

$$\frac{v_\mu}{c} = \frac{1 - \left(\frac{m_\pi}{m_\mu}\right)^2}{1 + \left(\frac{m_\pi}{m_\mu}\right)^2} \approx -0.270 \tag{44}$$

From here, we are home free. We can calculate γ_μ and the muon's kinetic energy first. It is convenient to remember that the electron mass is $511\,\mathrm{keV/c^2}$.

$$\gamma_\mu = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{(0.27c)^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.27^2}} \approx 1.0386 \tag{45}$$

$$\mathrm{KE}_\mu = (\gamma_\mu - 1)\, m_\mu c^2 = (1.0386 - 1)\,(207 m_{e^-})\, c^2 \tag{46}$$

$$= 0.0386\left(207 \cdot 511\,\mathrm{keV}/c^2\right) c^2 \approx 4.08 \times 10^6\,\mathrm{eV} = 4.08\,\mathrm{MeV} \tag{47}$$

Finally, we can calculate the energy of the antineutrino as well:

$$E_\nu = p_\nu c = -p_\mu c = -\gamma_\mu m_\mu v_\mu = -1.0386 \cdot \left(207 \cdot 5.11\,\mathrm{keV}/c^2\right) \cdot (-0.270c) \tag{48}$$

$$\approx 2.96 \times 10^7\,\mathrm{eV} = 29.6\,\mathrm{MeV} \tag{49}$$

11. The electromagnetic wave equation in one dimension is given by

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{50}$$

Prove that wave equation retains its form under Lorentz transformations. That is, show that

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t'^2} = 0 \tag{51}$$

where x, x', t, and t' are related by the Lorentz transformations. You may want to recall the chain rule: if $f = f(x', t')$, then

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x} \tag{52}$$

The point of this exercise is to show that the electromagnetic wave equation – derived from Maxwell's equations in free space – takes the same form in any reference frame. This amounts to demonstrating that (in free space anyway) Maxwell's equations are consistent with relativity. We start with the Lorentz transformations:

$$x' = \gamma\,(x + vt) \qquad t' = \gamma\left(t + \frac{vx}{c^2}\right) \tag{53}$$

If we have a wave function φ and wish to relate its derivatives in different reference frames, we can use the chain rule:

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \varphi}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial \varphi}{\partial t'}\frac{\partial t'}{\partial x} \qquad \text{and} \qquad \frac{\partial \varphi}{\partial t} = \frac{\partial \varphi}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial \varphi}{\partial x'}\frac{\partial x'}{\partial t} \tag{54}$$

We need to keep in mind that γ and v do not depend on the space or time variables, and may be treated as constants in the present problem. From the Lorentz transformations,

$$\frac{\partial x'}{\partial x} = \gamma \qquad \frac{\partial t'}{\partial x} = \gamma v/c^2 \qquad \frac{\partial t'}{\partial t} = \gamma \qquad \frac{\partial x'}{\partial t} = \gamma v \tag{55}$$

Thus,

$$\frac{\partial \varphi}{\partial x} = \gamma\frac{\partial \varphi}{\partial x'} + \frac{\gamma v}{c^2}\frac{\partial \varphi}{\partial t'} \qquad \text{and} \qquad \frac{\partial \varphi}{\partial t} = \gamma\frac{\partial \varphi}{\partial t'} + \gamma v\frac{\partial \varphi}{\partial x'} \tag{56}$$

Note that a more general approach would be to redefine the partial derivative operators:

$$\frac{\partial}{\partial x} = \gamma\frac{\partial}{\partial x'} + \frac{\gamma v}{c^2}\frac{\partial}{\partial t'} \qquad\qquad \frac{\partial}{\partial t} = \gamma\frac{\partial}{\partial t'} + \gamma v\frac{\partial}{\partial x'} \tag{57}$$

In any case, for the second derivatives, we simple repeat the procedure above.

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial x}\right) = \gamma\frac{\partial}{\partial x'}\left(\frac{\partial \varphi}{\partial x}\right) + \frac{v}{c^2}\frac{\partial}{\partial t'}\left(\frac{\partial \varphi}{\partial x}\right) = \gamma^2\frac{\partial^2 \varphi}{\partial x'^2} + \frac{2\gamma^2 v}{c^2}\frac{\partial^2 \varphi}{\partial x'\partial t'} + \frac{\gamma v^2}{c^4}\frac{\partial^2 \varphi}{\partial t'^2} \tag{58}$$

$$\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial t}\right) = \gamma\frac{\partial}{\partial t'}\left(\frac{\partial \varphi}{\partial t}\right) + \gamma v\frac{\partial}{\partial x'}\left(\frac{\partial \varphi}{\partial t}\right) = \gamma^2\frac{\partial^2 \varphi}{\partial t'^2} + 2\gamma^2 v\frac{\partial^2 \varphi}{\partial x'\partial t'} + \gamma^2 v^2\frac{\partial^2 \varphi}{\partial x'^2} \tag{59}$$

Substituting our results in the wave equation, and collecting terms:

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \frac{\partial^2\varphi}{\partial x'^2}\left(\gamma^2 - \frac{\gamma^2 v^2}{c^2}\right) + \frac{\partial^2\varphi}{\partial t'\partial x'}\left(\frac{2\gamma^2 v}{c^2} - \frac{2\gamma^2 v}{c^2}\right) + \frac{\partial^2\varphi}{\partial t'^2}\left(\frac{\gamma^2 v^2}{c^4} - \frac{\gamma^2}{c^2}\right) \tag{60}$$

$$= \frac{\partial^2\varphi}{\partial x'^2}\left(\gamma^2\right)\left(1 - \frac{v^2}{c^2}\right) + \frac{\partial^2\varphi}{\partial t'^2}\left(\frac{\gamma^2}{c^2}\right)\left(\frac{v^2}{c^2} - 1\right) \tag{61}$$

$$= \frac{\partial^2\varphi}{\partial x'^2}\left(\gamma^2\right)\left(\frac{1}{\gamma^2}\right) + \frac{\partial^2\varphi}{\partial t'^2}\left(\frac{\gamma^2}{c^2}\right)\left(\frac{-1}{\gamma^2}\right) = \frac{\partial^2\varphi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t'^2} \tag{62}$$

The wave equation is satisfied independent of reference frame. One may work this problem in reverse to *derive* the Lorentz transformations: by requiring that the wave equation be the same in all reference frames, one concludes that the Lorentz transformations must relate different reference frames.

A problem requiring this degree of detailed mathematics will not show up on the exam, as it is simply too tedious for the time allotted. In retrospect, I also realized that some of you have not seen partial derivatives yet, which means this problem would not be fair game anyway.