PH 253 Exam I Solutions

1. An electron and a proton are each accelerated starting from rest through a potential difference of 10.0 million volts (10^7 V). Find the momentum (in MeV/c) and kinetic energy (in MeV) of each, and compare the results with the classical expectation. Recall $\mathsf{PE} = \mathsf{q}\Delta\mathsf{V}$.

Solution: The key is conservation of energy. Each particle has a charge $|\mathsf{q}| = \mathsf{e}$, and when accelerated through a potential difference of $\Delta\mathsf{V}$ changes its potential energy by $\mathsf{e}\Delta\mathsf{V}$. This must equal the change in kinetic energy of the particle. Thus, $\mathsf{K} = \mathsf{e}\Delta\mathsf{V} = 10^7\,\mathrm{eV} = 1.6 \times 10^{-12}\,\mathrm{J}$ for both. From this, we can use the relativistic kinetic energy expression to find the velocity, and from that the momentum. The algebraic expressions are the same for both - the only difference between the two cases is the particle mass.

$$\mathsf{K} = \mathsf{q}\Delta\mathsf{V} = (\gamma - 1)\,\mathsf{mc}^2 \tag{1}$$

$$\gamma = \frac{\mathsf{q}\Delta\mathsf{V}}{\mathsf{mc}^2} + 1 \tag{2}$$

$$\mathsf{p} = \gamma\mathsf{mv} = \gamma\mathsf{m}\sqrt{1 - \frac{1}{\gamma^2}} \tag{3}$$

The last expression is one of convenience – we can just solve for γ this way, rather than bothering to solve for v. Makes no difference in the end, just saves a couple of lines of algebra. Classically, we would expect:

$$\mathsf{K}_{\mathsf{cl}} = \mathsf{q}\Delta\mathsf{V} = \frac{1}{2}\mathsf{mv}^2 \tag{4}$$

$$\mathsf{v}_{\mathsf{cl}} = \sqrt{\frac{2\mathsf{q}\Delta\mathsf{V}}{\mathsf{m}}} \tag{5}$$

$$\mathsf{p}_{\mathsf{cl}} = \mathsf{mv} = \sqrt{2\mathsf{mq}\Delta\mathsf{V}} \tag{6}$$

For the electron, we have:

$$\mathsf{K} = 10\,\mathrm{MeV} = 1.6 \times 10^{-12}\,\mathrm{J} \tag{7}$$

$$\mathsf{p} = 10.49\,\mathrm{MeV/c} = 5.6 \times 10^{-21}\,\mathrm{kg \cdot m/s} \tag{8}$$

$$\mathsf{K}_{\mathsf{cl}} = \mathsf{K} = 10\,\mathrm{MeV} \tag{9}$$

$$\mathsf{p}_{\mathsf{cl}} = 3.20\,\mathrm{MeV/c} = 1.71 \times 10^{-21}\,\mathrm{kg \cdot m/s} \tag{10}$$

For the proton, we have

$$\mathsf{K} = 10\,\mathrm{MeV} = 1.6 \times 10^{-12}\,\mathrm{J} \tag{11}$$

$$\mathsf{p} = 137\,\mathrm{MeV/c} = 7.34 \times 10^{-20}\,\mathrm{kg \cdot m/s} \tag{12}$$

$$\mathsf{K}_{\mathsf{cl}} = \mathsf{K} = 10\,\mathrm{MeV} \tag{13}$$

$$\mathsf{p}_{\mathsf{cl}} = 137\,\mathrm{MeV/c} = 7.31 \times 10^{-20}\,\mathrm{kg \cdot m/s} \tag{14}$$

For the heavier proton, a potential energy of $10\,\mathrm{MeV}$ only accelerates it about $\gamma \approx 1.01$, and classical mechanics works just fine. The same energy transferred to the much lighter electron accelerates it to $\gamma \approx 20.6$, well into the relativistic regime.

2. An electron is released from rest and falls under the influence of gravity. **(a)** How much power does it radiate? **(b)** How much energy is lost after it falls $1\,\mathrm{m}$? (*Hint:* $\mathsf{P}=\Delta\mathsf{K}/\Delta\mathsf{t}$, $\mathsf{y}=\frac{1}{2}\mathsf{g}\mathsf{t}^2$.)

Solution: The power emitted by a charge e with acceleration a is

$$\mathsf{P} = \frac{\mathsf{e}^2\mathsf{a}^2}{6\pi\epsilon_o\mathsf{c}^3} \tag{15}$$

In this case, under free fall the electron's acceleration is $\mathsf{g}\approx 9.81\,\mathrm{m/s^2}$, which gives

$$\mathsf{P} = \frac{\mathsf{e}^2\mathsf{g}^2}{6\pi\epsilon_o\mathsf{c}^3} \approx 5\times 10^{-52}\,\mathrm{W} \tag{16}$$

In a time t, starting from rest, an object under the influence of gravity falls a distance $\Delta\mathsf{y}=\frac{1}{2}\mathsf{g}\mathsf{t}^2$. Knowing the electron falls $\Delta\mathsf{y}=1\,\mathrm{m}$, the time it takes is

$$\mathsf{t} = \sqrt{\frac{2\Delta\mathsf{y}}{\mathsf{g}}} \approx 0.45\,\mathrm{s} \tag{17}$$

Since the power dissipation is constant, the energy lost is just power times time (since $\mathsf{P}=\Delta\mathsf{E}/\Delta\mathsf{t}$):

$$\Delta\mathsf{E} = \mathsf{P}\mathsf{t} = \frac{\mathsf{e}^2\mathsf{g}^2}{6\pi\epsilon_o\mathsf{c}^3}\sqrt{\frac{2\Delta\mathsf{y}}{\mathsf{g}}} \approx 2.5\times 10^{-52}\,\mathrm{J} \tag{18}$$

An utterly negligible amount. We don't need to worry about radiation of charges accelerated by gravity.

3. An electron initially moving at constant speed v is brought to rest with uniform deceleration a lasting for a time $\mathsf{t}=\mathsf{v}/\mathsf{a}$. Compare the electromagnetic energy radiated during this deceleration with the electron's initial kinetic energy. Express the ratio in terms of two lengths, the distance light travels in time t and the classical electron radius $\mathsf{r}_e=\mathsf{e}^2/4\pi\epsilon_o\mathsf{m}\mathsf{c}^2$.

Solution: The power emitted by a charge e with acceleration a is

$$\mathsf{P} = \frac{\mathsf{e}^2\mathsf{a}^2}{6\pi\epsilon_o\mathsf{c}^3} \tag{19}$$

In this case, we know that $\mathsf{a}=\mathsf{v}/\mathsf{t}$. The energy radiated in time t is just $\mathsf{U}=\mathsf{P}\mathsf{t}$, so

$$\mathsf{U} = \mathsf{P}\mathsf{t} = \frac{\mathsf{e}^2\mathsf{v}^2}{6\pi\epsilon_o\mathsf{c}^3\mathsf{t}} \tag{20}$$

The ratio of this energy to the kinetic energy before deceleration is

$$\frac{\mathsf{U}}{\mathsf{K}} = \frac{1}{\frac{1}{2}\mathsf{m}\mathsf{v}^2}\frac{\mathsf{e}^2\mathsf{v}^2}{6\pi\epsilon_o\mathsf{c}^3\mathsf{t}} = \frac{\mathsf{e}^2}{3\pi\epsilon_o\mathsf{m}\mathsf{c}^3\mathsf{t}} \tag{21}$$

Noting that the distance light travels in a time t is $\mathsf{r}_l=\mathsf{c}\mathsf{t}$ and using the expression for the classical electron radius above,

$$\frac{\mathsf{U}}{\mathsf{K}} = \frac{\mathsf{e}^2}{3\pi\epsilon_o\mathsf{m}\mathsf{c}^3\mathsf{t}} = \frac{\mathsf{e}^2}{4\pi\epsilon_o\mathsf{m}\mathsf{c}^2}\cdot\frac{4}{3}\cdot\frac{1}{\mathsf{c}\mathsf{t}} = \frac{4\mathsf{r}_e}{3\mathsf{r}_l} \tag{22}$$

4. Observer O notes that two events are separated in space and time by $600\,\mathrm{m}$ and $8\times 10^{-7}\,\mathrm{s}$. How fast must observer O' be moving relative to O in order that the events seem simultaneous?

Solution: For the events to be simultaneous for an observer in O', the time interval between them must be zero: $\Delta t' = 0$. We can relate the two time intervals with the Lorentz transformation:

$$\Delta t' = \gamma \left(\Delta t - \frac{v \Delta x}{c^2} \right) = 0 \tag{23}$$

$$\implies \quad \frac{v}{c} = \frac{c \Delta t}{\Delta x} \tag{24}$$

GIven $\Delta x = 600\,\text{m}$ and $\Delta t = 8 \times 10^{-7}\,\text{s}$, we find $v/c = 0.4$.

5. A bassist taps the lowest E on her bass at 140 beats per minute during one portion of a song. What tempo would an observer on a ship moving *toward* the bassist at $0.70c$ hear?

Solution: The time interval between taps for the stationary observer is

$$\Delta t = \frac{1\,\text{min}}{140\,\text{beats}} = \frac{3}{7}\,\text{sec/beat} \tag{25}$$

The moving observer sees a dilated time interval, longer by a factor $\gamma = 1/\sqrt{1 - v^2/c^2} \approx 1.4$:

$$\Delta t' = \gamma \Delta t \approx 0.6\,\text{sec/beat} \tag{26}$$

The tempo the moving observer sees is then the inverse of the time interval:

$$(\text{tempo})' = \frac{1}{\Delta t'} \approx 100\,\text{beats/min} \tag{27}$$

6. The proper lifetime of a certain particle is 100.0 ns. **(a)** How long does it live in the laboratory frame if it moves at $v = 0.960c$? **(b)** How far does it travel in the laboratory during that time? **(c)** What is the distance traveled in the lab according to an observer moving with the particle?

Solution: In the particle's own frame, a proper time of $\Delta t_p = 100\,\text{ns}$ passes. An observer in the laboratory (which we will denote as the primed frame) is in motion relative to the particle, and thus in motion with respect to the event of interest, the particle's decay. The lab observer therefore sees a dilated time interval:

$$\Delta t' = \gamma \Delta t_p = \frac{\Delta t_p}{\sqrt{1 - v^2/c^2}} \approx 357\,\text{ns} \tag{28}$$

According to the lab observer, then, the distance traveled is the time they observe times the relative velocity they observe (which both frames agree on):

$$\Delta x' = v \Delta t' = \gamma v \Delta t \approx 103\,\text{m} \tag{29}$$

For an observer traveling with the particle, the proper time of 100 ns passes, and the relative velocity is the same:

$$\Delta x = v \Delta t \approx 28.8\,\text{m} \tag{30}$$

7. Two electrons leave a radioactive sample in opposite directions, each having a speed $0.67c$ with respect to the sample. What is the speed of one electron relative to the other? That is, what would one electron say the other's speed is?

Solution: From the point of view of one of the electrons, it is sitting still and the other is moving away from it with the sum of the two electron's velocities. Thus, we must use the velocity addition formula. Let the electrons' velocities in the sample's reference frame be v_1 and v_2. The velocity of electron 2 from the reference frame of electron 1 (which we will denote as the 'primed' frame) is then:

$$v_2' = \frac{v_1 + v_2}{1 + v_1 v_2/c^2} = \frac{0.67c + 0.67c}{1 + (0.67c)(0.67c)/c^2} \approx 0.925c \tag{31}$$

8. A capacitor consists of two parallel rectangular plates with a vertical separation of $0.02\,\text{m}$. The east-west dimension of the plates is $0.2\,\text{m}$, the north-south dimension is $10\,\text{cm}$. The capacitor has been charged by connecting it temporarily to a battery of $300\,\text{V}$.

(a) How many excess electrons are on the negative plate?
(b) What is the electric field strength between the plates?

Now, give the quantities as they would be measured in a frame of reference which is moving eastward, relative to the laboratory in which the plates are at rest, with speed $0.6c$.

(c) The dimensions of the capacitor,
(d) The number of excess electrons on the negative plate,
(e) The electric field strength between the plates.

Solution: **(a)** The excess charge can be found from the definition of the capacitance and its specific form for two parallel plates:

$$C = \frac{Q}{\Delta V} = \frac{\epsilon_o A}{d} \quad \Longrightarrow \quad Q = \frac{\epsilon_o A \Delta V}{d} \approx 2.665^{-9}\,\text{C} \approx 1.66 \times 10^{10}\,\text{electrons} \tag{32}$$

Here ΔV is the potential difference applied to the battery, the area of the plates is the product of the east-west and north-south distance, $l_{ew} l_{ns}$, and d is the vertical separation.

(b) The electric field strength between the plates, treating them as infinite plates, can be found in two ways:

$$E = \frac{\Delta V}{d} = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_o A} = 15,000\,\text{V/m} \tag{33}$$

(c) Moving eastward, perpendicular to the direction separating the plates, we will have a contraction of the east-west length but *not* the north-south length or the separation. Thus, the new dimensions of the capacitor are

$$l_{ew}' = l_{ew}/\gamma = l_{ew}\sqrt{1 - v^2/c^2} = 0.16\,\text{m} \tag{34}$$
$$l_{ns}' = l_{ns} = 0.1\,\text{m} \tag{35}$$
$$d' = d = 0.02\,\text{m} \tag{36}$$

(d) The number of electrons per plate is the same, since charge is invariant.

(e) The electric field strength will increase, since we have the same number of electrons confined to effectively smaller plates. The area of the plates is now a factor of gamma smaller, $l_{ew}' l_{ns}' = l_{ew} l_{ns}/\gamma$, meaning the

charge density is a factor γ higher, and thus the electric field is also a factor of γ higher:

$$E' = \frac{\sigma'}{\epsilon_o} = \frac{\gamma\sigma}{\epsilon_o} = \gamma E = 18,750 \, \text{V/m} \tag{37}$$

Constants:

$$g \approx 9.81 \,\text{m/s}^2$$

$$N_A = 6.022 \times 10^{23} \,\text{things/mol}$$

$$k_e \equiv 1/4\pi\epsilon_o = 8.98755 \times 10^9 \,\text{N} \cdot \text{m}^2 \cdot \text{C}^{-2}$$

$$\epsilon_o = 8.85 \times 10^{-12} \,\text{C}^2/\text{N} \cdot \text{m}^2$$

$$\mu_0 \equiv 4\pi \times 10^{-7} \,\text{T} \cdot \text{m/A}$$

$$e = 1.60218 \times 10^{-19} \,\text{C}$$

$$h = 6.6261 \times 10^{-34} \,\text{J} \cdot \text{s} = 4.1357 \times 10^{-15} \,\text{eV} \cdot \text{s}$$

$$\hbar = \frac{h}{2\pi}$$

$$k_B = 1.38065 \times 10^{-23} \,\text{J} \cdot \text{K}^{-1} = 8.6173 \times 10^{-5} \,\text{eV} \cdot \text{K}^{-1}$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.99792 \times 10^8 \,\text{m/s}$$

$$hc = 1240 \,\text{eV} \cdot \text{nm}$$

$$m_e = 9.10938 \times 10^{-31} \,\text{kg} \qquad m_e c^2 = 510.998 \,\text{keV}$$

$$m_p = 1.67262 \times 10^{-27} \,\text{kg} \qquad m_p c^2 = 938.272 \,\text{MeV}$$

Quadratic formula:

$$0 = ax^2 + bx^2 + c \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Basic Equations:

$$\vec{E} = \sigma/\epsilon_0 \ \text{capacitor}$$

$$C = \epsilon_0 A/d$$

$$\vec{F}_{\text{net}} = m\vec{a} \ \text{Newton's Second Law}$$

$$\vec{F}_{\text{centr}} = -\frac{mv^2}{r}\hat{r} \ \text{Centripetal}$$

$$\vec{F}_{12} = k_e \frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} = q_2 \vec{E}_1 \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

$$\vec{E}_1 = \vec{F}_{12}/q_2 = k_e \frac{q_1}{r_{12}^2}\hat{r}_{12}$$

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

$$0 = ax^2 + bx^2 + c \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

E & M

$$\vec{F}_{12} = k_e \frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} = q_2 \vec{E}_1 \qquad \vec{r}_{12} = \vec{r}_1 - \vec{r}_2$$

$$\vec{E}_1 = \vec{F}_{12}/q_2 = k_e \frac{q_1}{r_{12}^2}\hat{r}_{12}$$

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

Oscillators & waves

$$E = \frac{1}{2}kA^2 = \frac{1}{2}\omega^2 m A^2 = 2\pi^2 m f^2 A^2$$

$$\omega = 2\pi f = \sqrt{k/m}$$

$$c = \lambda f$$

Approximations, $x \ll 1$

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n+1)x^2 \qquad e^x \approx 1 + x + \frac{1}{2}x$$

$$\sin x \approx x - \frac{1}{6}x^3 \qquad \cos x \approx 1 - \frac{1}{2}x^2$$

Radiation

$$P_{\text{rad}} = \frac{q^2 a^2}{6\pi\epsilon_o c^3} \qquad \text{total emitted power, E and B fields}$$

Relativity

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Delta t'_{\text{moving}} = \gamma \Delta t_{\text{stationary}} = \gamma \Delta t_p$$

$$L'_{\text{moving}} = \frac{L_{\text{stationary}}}{\gamma} = \frac{L_p}{\gamma}$$

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \qquad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$v_{\text{obj}} = \frac{v + v'_{\text{obj}}}{1 + \frac{vv'_{\text{obj}}}{c^2}} \qquad v'_{\text{obj}} = \frac{v_{\text{obj}} - v}{1 - \frac{vv_{\text{obj}}}{c^2}}$$

$$KE = (\gamma - 1)mc^2 = \sqrt{m^2 c^4 + c^2 p^2} - mc^2$$

$$E_{\text{rest}} = mc^2$$

$$p = \gamma m v$$

$$E^2 = p^2 c^2 + m^2 c^4 = (\gamma mc^2)^2$$

Calculus of possible utility:

$$\int \frac{1}{x}dx = \ln x + c$$

$$\int u\,dv = uv - \int v\,du$$

Vectors:

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2} \quad \text{magn} \qquad \theta = \tan^{-1}\left[\frac{F_y}{F_x}\right] \quad \text{dir}$$