PH 253 Exam 2

Instructions

1. Solve four of the six problems below. All problems have equal weight.
2. Do your work on separate sheets.
3. You are allowed 1 sheet of standard 8.5x11 in paper and a calculator.

1. A particle of mass m moves in one dimension under the influence of a potential energy function

$$V(x) = \begin{cases} \infty & x \leqslant 0 \\ -\frac{\beta}{x} & x > 0 \end{cases} \tag{1}$$

where β is a positive constant. One of the wavefunctions for the particle (i.e., a solution to Schrödinger's equation) is given by

$$\psi(x) = Axe^{-\alpha x} \quad x > 0 \tag{2}$$

where A and α are constants. Find the energy of the particle in this state in terms of $\hbar$, m, and β only. *Hint: you may not need to normalize. Try Schrödinger's equation.*

2. A particle of mass m is trapped in a one dimensional oscillator potential, $V(x) = \frac{1}{2}m\omega^2 x^2$ for $-\infty < x < \infty$. The wave functions for the first two states are shown below.

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} \qquad \psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \sqrt{\frac{2m\omega}{\hbar}}\, x\, e^{-m\omega x^2/2\hbar} \tag{3}$$

(a) Find the energy of the ψ_0 state.
(b) What are $\langle x \rangle$ and $\langle p \rangle$ for each state? An explicit calculation is unnecessary if you can provide a physical explanation for your result.

To save you some time, we note $\frac{d}{dx}(e^{-x^2/2a^2}) = -\frac{x}{a^2}e^{-x^2/2a^2}$ and $\frac{d^2}{dx^2}(e^{-x^2/2a^2}) = \frac{x^2-a^2}{a^4}e^{-x^2/2a^2}$. Make use of the integral table on the formula sheet.

3. At $t=0$, the (normalized) wave function of a free particle of mass m is given by

$$\psi(x,0) = \begin{cases} \frac{1}{\sqrt{x_2-x_1}} e^{ik_o x} & x_1 \leqslant x \leqslant x_2 \\ 0 & x < x_1 \quad \text{and} \quad x > x_2 \end{cases} \tag{4}$$

(a) Find $\langle x \rangle$ at $t=0$. Your answer should agree with your commonsense expectation.
(b) What is the probability that the particle is found in the interval $[x_1, \frac{1}{2}(x_1+x_2)]$?

4. Given the wave function

$$\psi(x) = \begin{cases} Ne^{\kappa x} & x < 0 \\ Ne^{-\kappa x} & x > 0 \end{cases} \tag{5}$$

(a) Find N needed to normalize ψ, and **(b)** find $\langle x \rangle$, $\langle x^2 \rangle$, and Δx.

5. A hydrogen atom has a radius of ~0.05 nm. **(a)** Assuming we know the position of an electron in a hydrogen atom to an accuracy of 1% of this radius, estimate the uncertainty in the velocity of the electron. **(b)** How does this value compare to the Bohr model's estimate of the velocity of the electron in the ground state? (*Hint: see formula sheet.*)

6. A collection of hydrogen atoms in the ground state is illuminated with ultraviolet light of wavelength 59.0 nm. Find the kinetic energy of the emitted electrons.

Constants:

$$k_e \equiv 1/4\pi\epsilon_o = 8.98755 \times 10^9\,\mathrm{N \cdot m^2 \cdot C^{-2}}$$
$$\epsilon_o = 8.85 \times 10^{-12}\,\mathrm{C^2/N \cdot m^2}$$
$$e = 1.60218 \times 10^{-19}\,\mathrm{C}$$
$$h = 6.6261 \times 10^{-34}\,\mathrm{J \cdot s} = 4.1357 \times 10^{-15}\,\mathrm{eV \cdot s}$$
$$\hbar = \frac{h}{2\pi} \qquad hc = 1239.84\,\mathrm{eV \cdot nm}$$
$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.99792 \times 10^8\,\mathrm{m/s}$$
$$m_e = 9.10938 \times 10^{-31}\,\mathrm{kg} \qquad m_e c^2 = 510.998\,\mathrm{keV}$$
$$m_p = 1.67262 \times 10^{-27}\,\mathrm{kg} \qquad m_p c^2 = 938.272\,\mathrm{MeV}$$
$$m_n = 1.67493 \times 10^{-27}\,\mathrm{kg} \qquad m_n c^2 = 939.565\,\mathrm{MeV}$$

Schrödinger

$$i\hbar \frac{d\Psi}{dt} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi + V(x)\Psi \quad \text{1D time-dep}$$
$$E\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi + V(x)\psi \quad \text{1D time-indep}$$
$$\int_{-\infty}^{\infty} |\psi(x)|^2\,dx = 1 \qquad P(\text{in } [x, x+dx]) = |\psi(x)|^2 \qquad \text{1D}$$
$$\int_0^{\infty} |\psi(r)|^2\,4\pi r^2\,dr = 1 \qquad P(\text{in } [r, r+dr]) = 4\pi r^2 |\psi(r)|^2 \qquad \text{3D}$$
$$\langle x^n \rangle = \int_{-\infty}^{\infty} x^n P(x)\,dx \qquad \text{1D} \qquad \langle r^n \rangle = \int_0^{\infty} r^n P(r)\,dr \qquad \text{3D}$$
$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \left(-i\hbar \frac{d}{dx}\right)\psi\,dx$$
$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \qquad \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

Oscillators

$$E = \left(n + \frac{1}{2}\right)hf$$
$$E = \frac{1}{2}kA^2 = \frac{1}{2}\omega^2 mA^2 = 2\pi^2 mf^2 A^2$$
$$\omega = 2\pi f = \sqrt{k/m}$$

Approximations, $x \ll 1$

$$(1+x)^n \approx 1 + nx + \frac{1}{2}n(n+1)x^2 \qquad \tan x \approx x + \frac{1}{3}x^3$$
$$e^x \approx 1 + x + \frac{1}{2}x \qquad \sin x \approx x - \frac{1}{6}x^3 \qquad \cos x \approx 1 - \frac{1}{2}x^2$$

Misc Quantum

$$E = hf \qquad p = h/\lambda = E/c \qquad \lambda f = c \qquad \text{photons}$$
$$\lambda = \frac{h}{|\vec{p}|} = \frac{h}{\gamma m v} \approx \frac{h}{mv}$$
$$\Delta x \Delta p \geqslant \frac{h}{4\pi} \qquad \Delta E \Delta t \geqslant \frac{h}{4\pi}$$
$$|\psi| = \psi^* \psi \qquad |e^{ikx}| = 1$$

Bohr

$$\begin{aligned} E_n &= -13.6\,\mathrm{eV}/n^2 \qquad \text{Hydrogen} \\ E_i - E_f &= -13.6\,\mathrm{eV}\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) = hf \\ L = mvr &= n\hbar \\ v^2 &= \frac{n^2\hbar^2}{m_e^2 r^2} = \frac{k_e e^2}{m_e r} \end{aligned}$$

Calculus of possible utility:

$$\int \frac{1}{x}\,dx = \ln x + c$$
$$\int u\,dv = uv - \int v\,du$$
$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$
$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$
$$\int e^{-ax}\,dx = -\frac{1}{a}e^{-ax} + C$$
$$\int_0^{\infty} x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}$$
$$\int_0^{\infty} xe^{-ax^2}\,dx = \frac{1}{2a}$$
$$\int_0^{\infty} x^2 e^{-ax^2}\,dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}}$$
$$\int_{-\infty}^{\infty} x^3 e^{-ax^2}\,dx = \int_{-\infty}^{\infty} xe^{-ax^2}\,dx = 0$$
$$\int_0^{\infty} x^4 e^{-ax^2}\,dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}}$$