University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair — Spring 2019

Exam 1 Solution

1. A stick of length L is at rest in a system O and is oriented at an angle θ with respect to the x axis. An observer in system O' travels at velocity v with respect to the system O along the x axis. On a recent homework problem, you were asked for the apparent angle θ' that the stick makes with the x' axis according to the observer in O'. What is the apparent *length* L' that the observer in O' sees, in terms of given quantities? The x and x' axes are parallel.

Solution: Let the reference frame at rest with respect to the stick be the 'unprimed' frame, with the primed frame corresponding to the observer moving at speed v relative to the stick. Since the relative motion is along the (presumed collinear) x and x' axes, the primed observer sees distances along the x' axis as contracted relative to the reference frame of the stick.

In the stick's (unprimed) frame, the horizontal extent of the stick along the x axis is $L_x = L\cos\theta$, while the extent along the y axis is $L_y = L\sin\theta$. For the moving observer, the x dimensions are contracted, but not the y, and thus

$$\begin{aligned} L'_x &= L_x/\gamma = L_x\sqrt{1-\frac{v^2}{c^2}} \\ L'_y &= L_y \end{aligned} \tag{1}$$

The stationary observer sees the stick as having length $L = \sqrt{L_x^2 + L_y^2}$. The moving observer sees the stick as having a length

$$L' = \sqrt{(L'_x)^2 + \left(L'_y\right)^2} = \sqrt{L_x^2\left(1-\frac{v^2}{c^2}\right) + L_y^2} = L\sqrt{1-\frac{v^2}{c^2}\cos^2\theta} = L\sqrt{\frac{L_x^2}{\gamma^2} + L_y^2} \tag{2}$$

$$L' = L\sqrt{\frac{\cos^2\theta}{\gamma^2} + \sin^2\theta} \tag{3}$$

2. An atomic clock aboard a spaceship runs slow compared with an earth-based atomic clock at a rate of 1.0 second per day. What is the speed of the spaceship? Note for convenience $60 \times 60 \times 24 = 8.64 \times 10^4$.

Solution: The proper time t_p is that measured on earth, while a dilated time $t = \gamma t_p$ is measured on the ship. If the clock aboard the ship is 1 s per day slow, then using 1 day = 86400 s

$$t - t_p = \gamma t_p - t_p = (\gamma - 1)\, t_p = (\gamma - 1)\,(86400\,\text{s}) = 1\,\text{s} \tag{4}$$

This gives

$$\gamma = \frac{1}{86400} + 1 = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{5}$$

Solving for v, we find $v \approx 0.0048c$.

3. At time $t_1 = 0$, a boiler explodes in town A. At time $t_2 = 0.0003\,\text{s}$ a similar boiler explodes in town B, 150 km away from the first explosion. Show that, in a reference frame of a spaceship moving at speed greater than 0.60 c from town A toward town B, the first explosion occurs *after* the second.

Solution: Let the ground be the unprimed frame. In this frame, the time between events $\Delta t = t_2 - t_1$ is positive. Transforming to a primed frame aboard the ship,

$$\Delta t' = \gamma\left(\Delta t - \frac{vx}{c^2}\right) \tag{6}$$

where $x = 150\,\text{km}$ is the distance between the events and v is the speed of the ship. We want to find the speed v such that $\Delta t'$ is negative so the order of events is reversed. Thus,

$$\Delta t' = \gamma\left(\Delta t - \frac{vx}{c^2}\right) > 0 \tag{7}$$

$$\Delta t > \frac{vx}{c^2} \tag{8}$$

$$\frac{v}{c} = \frac{c\Delta t}{x} \approx 0.6 \tag{9}$$

Thus, for speeds $v > 0.6c$, the order of events is reversed, and at $v = 0.6c$ the events happen simultaneously.

4. An electron (charge e and mass m_e) moves in a solid at a thermal speed of $v_o = 10^5\,\text{m/s}$ ($v \ll c$), and after colliding with an atom it experiences a constant deceleration such that it comes to rest after traveling $d = 3 \times 10^{-9}\,\text{m}$. What fraction of the electron's initial kinetic energy is lost as radiation during the deceleration? *Hint: recall from kinematics how to relate time, distance, and acceleration.*

5. Start with kinematics to find the acceleration in terms of the initial velocity and distance:

$$d = v_o t - \frac{1}{2}at^2 = \frac{v_o^2}{a} - \frac{1}{2}a\frac{v_o^2}{2a} \tag{10}$$

$$\Longrightarrow \qquad a = \frac{v_o^2}{d} \tag{11}$$

We also know $v_f = 0 = v_o - at$, so $|t| = v_o/a$:

$$|t| = \frac{v_o}{a} = \frac{2v_o d}{v_o^2} = \frac{2d}{v_o} \tag{12}$$

From the Larmor formula we get the radiated power P, and from that the radiated energy $E = Pt$.

$$P = \frac{q^2a^2}{6\pi\epsilon_o c^3} = \frac{q^2v_o^4}{6\pi\epsilon_o c^3 \cdot 4d^2} \tag{13}$$

The desired ratio is $E/K = Pt/K$. With the electron's charge being $q = e$,

$$\frac{E}{K} = Pt \cdot \frac{1}{\frac{1}{2}mv_o^2} = \frac{e^2v_o^4}{6\pi\epsilon_o c^3 \cdot 4d^2} \cdot \frac{2d}{v_o} \cdot \frac{1}{\frac{1}{2}m_e v_o^2} = \frac{q^2 v_o}{6\pi\epsilon_o c^3 m_e d} \approx 2 \times 10^{-10} \tag{14}$$

6. (a) Show that the momentum of a particle can be expressed in the concise form $p = Ev/c^2$. **(b)** Given the following:

$$p = \gamma m v = \frac{mv}{\sqrt{1 - v^2/c^2}}$$
$$E = \gamma m c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

demonstrate that $E^2 = p^2c^2 + m^2c^4$.

Solution: Part a:

$$p = \gamma m v = \gamma m \left(c^2 \frac{1}{c^2} \right) v = \gamma m c^2 \left(\frac{v}{c^2} \right) = E \frac{v}{c^2} \tag{15}$$

Part b:

$$p^2c^2 + m^2c^4 = \frac{m^2v^2c^2}{1 - v^2/c^2} + m^2c^4 = \frac{m^2v^2c^2}{1 - v^2/c^2} + \frac{m^2c^4\left(1 - v^2/c^2\right)}{1 - v^2/c^2} = \frac{m^2v^2c^2 + m^2c^4 - m^2c^2v^2}{1 - v^2/c^2} \tag{16}$$

$$p^2c^2 + m^2c^4 = \frac{m^2c^4}{1 - v^2/c^2} = \left(mc^2\right)^2 \gamma^2 = \left(\gamma m c^2\right)^2 = E^2 \tag{17}$$

$$\implies \qquad E = \gamma m c^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{18}$$