University of Alabama
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Problem Set 1: Solutions

1. A classic "paradox" involving length contraction and the relativity of simultaneity is as follows: Suppose a runner moving at $0.75c$ carries a horizontal pole 15 m long toward a barn that is 10 m long. The barn has front and rear doors. An observer on the ground can instantly and simultaneously open and close the two doors by remote control. When the runner and the pole are inside the barn, the ground observer closes and then opens both doors so that the runner and pole are momentarily captured inside the barn and then proceed to exit the barn from the back door. Do both the runner and the ground observer agree that the runner makes it safely through the barn?

Solution: Given the relative velocity between the reference frames $v = 0.75c$, we will have to account for length contraction and the relativity of simultaneity. One point to make clear: in the barn's reference frame, the doors close and then open immediately, and at the same time. They do not stay closed.

From the point of view of the person in the barn, the pole is moving toward them at velocity $v = 0.75c$, and its length is contracted by a factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1.51 \tag{1}$$

Thus, instead of 15 m the person in the barn sees the pole as having a length $15/\gamma \approx 9.9\,\text{m}$, and observes the runner to make it safely through.

From the point of view of the runner, the barn is in motion, and appears shortened by a factor γ to $10/\gamma \approx 6.6\,\text{m}$. When do the doors close from the runner's point of view? The runner will not see the doors close at the same time, since the runner is in motion relative to the doors and they are spatially separated. If the person in the barn sees the doors closing with time delay $\Delta t = 0$ (i.e., simultaneously), the runner sees a time delay $\Delta t'$ governed by the Lorentz transformation:

$$\Delta t' = \gamma \left(\Delta t + \frac{v \Delta x}{c^2} \right) \tag{2}$$

Here Δx is the separation between events – the front door closing and the back door closing – in the barn's reference frame, i.e., the length of the barn L_B. If we call the front door opening the first event and the back door closing the second event, then we are implying the runner's heading is positive, and that makes the distance Δx negative. This makes Δx negative, which means that the runner sees the *front door close first* and then the back:

$$\Delta t' = -\frac{\gamma v L_B}{c^2} \approx 37.8\,\text{ns} \tag{3}$$

From the person in the barn's point of view, he will see the runner reach the front door, then close it immediately. At that point, the runner also thinks the pole is at the front door, and that part of its 15 m length (according to him) is still sticking out of the back of the barn (which is 6.6 m long according to the runner). The runner then has time $\Delta t'$ to get that much of the pole in through the back door before it closes. The runner doesn't need to get out of the barn completely – remember, the doors will close and then open again immediately, the runner just can't be caught in the middle of either door while that happens. So, the runner must go the length of the pole minus the length of the barn before $\Delta t'$ passes. That length is:

$$L'_p - L'_B = 15\,\text{m} - \frac{10\,\text{m}}{\gamma} \approx 8.38\,\text{m} \tag{4}$$

At velocity v, going this length takes $8.38\,\text{m}/0.75c \approx 37.2\,\text{ns}$, leaving 0.6 ns to spare before the rear door closes. Thus, both the runner and the observer in the barn agree that the runner makes it through. Thus, both observers agree that the pole does not smash into the doors, but makes it through the barn safely.

We have only given a quick discussion here, one can perform a much more rigorous analysis. For further discussion see, for example:

`http://hyperphysics.phy-astr.gsu.edu/hbase/relativ/polebarn.htmll`
`http://en.wikipedia.org/wiki/Ladder_paradox`
`http://www.xs4all.nl/~johanw/PhysFAQ/Relativity/SR/barn_pole.html`

And an applet for good measure:

`http://webphysics.davidson.edu/physlet_resources/special_relativity/ex1.html`

(The links should be clickable.)

2. A pilot is supposed to fly due east from A to B and then back again to A due west. The velocity of the plane in air is u' and the velocity of the air with respect to the ground is v. The distance between A and B is l and the plane's air speed u' is constant.

(a) If $v=0$ (still air) show that the time for the round trip is $t_o = 2l/u'$.
(b) Suppose that the air velocity is due east (or west). Show that the time for a round trip is then

$$t_E = \frac{t_o}{1 - v^2/\left(u'\right)^2}$$

(c) Suppose the air velocity is due north (or south). Show that the time for a round trip is then

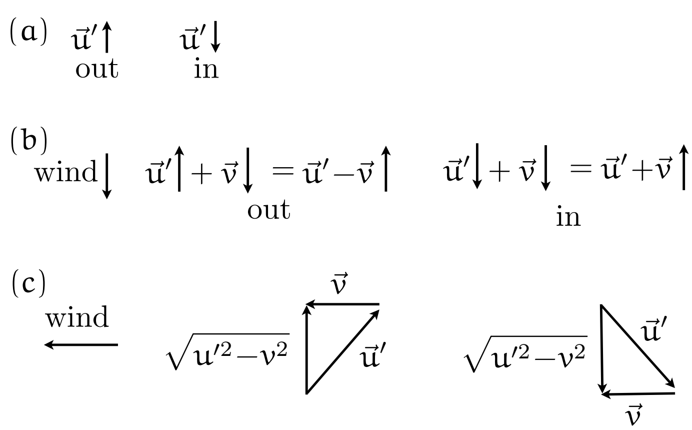
$$t_N = \frac{t_o}{\sqrt{1 - v^2/\left(u'\right)^2}}$$

(d) In parts (b) and (c) we must assume that $v < u'$. Why?

Solution: Though this problem does not involve relativity at all, classical problems involving relative velocities look strikingly similar.

Find / Given: We are to find the round-trip transit time for the plane with wind speed in various directions. We are given the plane's speed and the wind's speed and direction.

Sketch: In part (a) we simply have a trip of total distance $2l$ at speed u'. In part (b) we have the wind's velocity with and against that of the plane, and must add and subtract the two velocities to get the net velocity. In part (c) we have the net velocity at a right angle to the wind, and must find it by vector decomposition.



Relevant equations: Since the situation involves motion at constant velocity, we need only the relationship between distance (l), net velocity (v), and time (t): $t = l/v$. Additionally, we will need the rules for addition of vectors.

Symbolic solution: In all three cases, we may find the time for the outward trip (t_{out}) and inward trip (t_{in}) separately and add them together to get the total round-trip time. For each leg, we divide the distance covered l by the net forward velocity.

(a) For each leg of the trip we cover distance l at velocity v:

$$t_{out} = \frac{l}{u'} \tag{5}$$

$$t_{in} = \frac{l}{u'} \tag{6}$$

$$t_{net} = \frac{l}{u'} + \frac{l}{u'} = \frac{2l}{u'} \equiv t_o \tag{7}$$

(b) Suppose the wind velocity is due east. On the outward trip, we are going with the wind, so the net forward velocity is that of the plane plus that of the wind. On the inward trip, going westward, we are going against the wind, and the net forward velocity is the plane's velocity minus that of the wind.[i] Thus,

$$t_{out} = \frac{l}{u' + v} \tag{8}$$

$$t_{in} = \frac{l}{u' - v} \tag{9}$$

$$t_{net} = \frac{l}{u' + v} + \frac{l}{u' - v} = \frac{l\,(u' - v) + l\,(u' - v)}{(u' + v)\,(u' - v)} = \frac{2lu'}{u'^2 - v^2} = \frac{2l}{u'}\frac{1}{1 - v^2/u'^2} = \frac{t_o}{1 - v^2/u'^2} \tag{10}$$

(c) If the plane is to go forward with a side-to-side wind, the pilot must steer into the wind as shown in the sketch. This will reduce the net forward progress, and the net forward velocity is given by the vector diagram in the sketch, $\sqrt{u'^2 - v^2}$. The velocity is the same in both outward and inward trips, so we may just calculate one of the trips and double the result:

$$t_{net} = 2t_{out} = \frac{2l}{\sqrt{u'^2 - v^2}} = \frac{2l}{\sqrt{u'^2\,((1 - v^2/u'^2)}} = \frac{2l}{u'\sqrt{1 - v^2/u'^2}} = \frac{t_o}{\sqrt{1 - v^2/u'^2}} \tag{11}$$

(d) In part (b), we must assume the wind's velocity is smaller than that of the plane for two reasons. First, mathematically the inward trip time would be negative, which is unphysical. Second, if the wind speed were higher than that of the plane, it would not be able to make *any* forward progress to ever complete the outward trip! In part (c), the vector diagram makes it clear that if the wind speed were larger than the plane's speed, no forward progress could be made. Mathematically, the net forward velocity would be negative in this case ... which also means that the plane simply can't

[i]If we chose the wind going westward, the result would be the same.

go forward.

Numeric solution: n/a.

Double check: First, we can check that in all cases we have the correct units: distance divided by velocity gives time. Second, if we set the wind speed to zero for parts (b) and (c) we should recover the original result from part (a).

3. The length of a spaceship is measured to be exactly half its proper length. **(a)** What is the speed of the spaceship relative to the observer's frame? **(b)** What is the dilation of the spaceship's unit time?

Solution: This problem involves only time dilation and length contraction.

Find / Given: We are to find the speed of the spaceship and the dilation factor of the ship's time given the ratio of the length of the ship in its own reference frame to that seen by a moving observer.

Sketch: n/a.

Relevant equations: For the first portion, we can relate the ratio of the ship's length in its own frame (L_p, its proper length) to that measured by the moving observer (L') through the length contraction formula: $L' = L_p/\gamma$. With the definition of γ below, we can find the ship's velocity v. For the second portion, the time dilation formula relates the elapsed time according to the moving observer $\Delta t'$ to that of the ship Δt: $\Delta t' = \gamma \Delta t$.

$$L' = L_p/\gamma \tag{12}$$

$$\Delta t' = \gamma \Delta t \tag{13}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{14}$$

Symbolic solution: Since the moving observer sees the ship at half its rest length, the proper length is twice as big as that measured by the observer, and $\gamma = L_p/L' = 2$.

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{15}$$

$$\gamma^2 = \frac{1}{1 - v^2/c^2} \tag{16}$$

$$1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2} \tag{17}$$

$$\frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2} \tag{18}$$

The time dilation factor is the ratio of the observer's time to that of the ship, which is γ

Numeric solution: For the first part, $\frac{v}{c} = \frac{\sqrt{3}}{2} \approx 0.866$. For the second part, $\gamma = 2$ is the dilation factor.

Double check: If we let $\gamma \to 1$, we should recover the classical non-relativistic result. In the first part, that implies that $L_p = L'$, as expected, and in the second it implies that the dilation factor is zero, as expected. We also note that the formula for v/c is dimensionless in the first part, as is the formula for the dilation factor in the second. Finally, our velocities are less than c, so there is no physical impossibility here.

4. Derive the relativistic acceleration transformation

$$a'_x = \frac{a_x \left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 - \frac{u_x v}{c^2}\right)^3}$$

in which $a_x = du_x/dt$ and $a'_x = du'_x/dt'$. *Hint:* $du'_x/dt' = (du'_x/dt)\,(dt/dt')$

Solution: This problem teaches us a nice lesson: what is convenient for everyday physics is hideous and ugly relativistically. That isn't because relativity is ugly – it is because our everyday definitions of things like acceleration are faulty in the domain of relativity. One can define proper velocities and accelerations which transform nicely, but unfortunately those quantities are a bit annoying for everyday situations.

Find / Given: We are to derive the relativistic acceleration transformation, armed with little more than the Lorentz transformations and calculus.

Sketch: n/a.

Relevant equations: We need the Lorentz transformation for time, the velocity addition rule in one dimension, and the chain rule:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \tag{19}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{20}$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{21}$$

Symbolic solution: The acceleration in the primed frame is $a'_x = du'_x/dt'$, while in the unprimed frame it is du_x/dt. The difficulty here is that we must transform both the velocity and time parts, and this is where the chain rule comes in handy:

$$a'_x = \frac{du'_x}{dt'} = \frac{du'_x/dt}{dt'/dt} \tag{22}$$

The numerator and denominator can now be found from the velocity addition rule and the time transformation, respectively. We must take care that u_x varies in time but v does not. Noting that $du_x/dt = a_x$, and handling the numerator first,[ii]

$$\frac{du'_x}{dt} = \frac{d}{dt}\left(\frac{u_x - v}{1 - \frac{u_x v}{c^2}}\right) = \frac{a_x}{1 - \frac{u_x v}{c^2}} + \frac{(u_x - v)(-1)\left(-\frac{a_x v}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2} \tag{23}$$

$$= \frac{a_x\left(1 - \frac{u_x v}{c^2}\right)}{1 - \frac{u_x v}{c^2}} + \frac{(u_x - v)\left(\frac{a_x v}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2} = \frac{a_x - a_x\frac{u_x v}{c^2} + a_x\frac{u_x v}{c^2} - a_x\frac{v^2}{c^2}}{\left(1 - \frac{u_x v}{c^2}\right)^2} \tag{24}$$

$$= \frac{a_x\left(1 - \frac{v^2}{c^2}\right)}{\left(1 - \frac{u_x v}{c^2}\right)^2} \tag{25}$$

Whew! Now for the denominator (noting that since v is constant so is γ):

$$\frac{dt'}{dt} = \frac{d}{dt}\left[\gamma\left(t - \frac{vx}{c^2}\right)\right] = \gamma\left(1 - \frac{a_x v}{c^2}\right) \tag{26}$$

Putting it all together, and recalling $\gamma = 1/\sqrt{1 - v^2/c^2}$:

$$a'_x = \frac{du'_x}{dt'} = \frac{du'_x/dt}{dt'/dt} = \frac{\frac{a_x\left(1-\frac{v^2}{c^2}\right)}{\left(1-\frac{u_x v}{c^2}\right)^2}}{\gamma\left(1 - \frac{a_x v}{c^2}\right)} = \frac{a_x\left(1 - \frac{v^2}{c^2}\right)}{\gamma\left(1 - \frac{u_x v}{c^2}\right)^3} = \frac{a_x\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 - \frac{u_x v}{c^2}\right)^3} \tag{27}$$

Numeric solution: n/a.

[ii] Also remember that $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'}{g} - \frac{fg'}{g^2}$.

Double check: We have mathematically proven the desired result, no additional checks are necessary.

5. A charge q at $x=0$ accelerates from rest in a uniform electric field $\vec{E}$ which is directed along the positive x axis.

(a) Show that the acceleration of the charge is given by

$$a = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

(b) Show that the velocity of the charge at any time t is given by

$$v = \frac{qEt/m}{\sqrt{1 + (qEt/mc)^2}}$$

(c) Find the distance the charge moves in a time t. *Hint:* `http://integrals.wolfram.com`

Solution: I changed the notation in the problem slightly since I found it annoying after the fact – just v for velocity and a for acceleration, since it is a 1D problem anyway.

Find/given: We are to find the acceleration, velocity, and position as a function of time for a particle in a uniform electric field. We are given the electric force and the boundary conditions $x=0$, $v=0$ at $t=0$.

Sketch: n/a.

Relevant equations: We will need only $F = dp/dt$, $p = \gamma m v$, the definition of γ (given in previous problems), and a good knowledge of calculus (including the chain rule given in the last problem).

Symbolic solution: First, we must relate force and acceleration relativistically. Since velocity is explicitly a function of time here, so is γ, and we must take care.

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = \gamma m \frac{dv}{dt} + m v \frac{d\gamma}{dt} \tag{28}$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-v^2}c^2} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{2v}{c^2}\right)\frac{dv}{dt}}{\left(1-v^2/c^2\right)^{3/2}} = \frac{v}{c^2}\frac{1}{\left(1-v^2/c^2\right)^{3/2}} \tag{29}$$

$$F = \gamma m \frac{dv}{dt} + \frac{mv^2}{c^2}\frac{1}{\left(1-v^2/c^2\right)^{3/2}}\frac{dv}{dt} = ma\left(\frac{1}{\sqrt{1-v^2/c^2}} + \frac{\frac{v^2}{c^2}}{\left(1-v^2/c^2\right)^{3/2}}\right) \tag{30}$$

$$F = \frac{ma}{\left(1-v^2/c^2\right)^{3/2}} \tag{31}$$

That accomplished, we can set the net force equal to the electric force qE and solve for acceleration:

$$F = qE = \frac{ma}{\left(1-v^2/c^2\right)^{3/2}} \tag{32}$$

$$a = \frac{qE}{m}\frac{1}{\left(1-v^2/c^2\right)^{3/2}} \tag{33}$$

We can find velocity by writing a as dv/dt and noticing that the resulting equation is separable.

$$a = \frac{dv}{dt} = \frac{qE}{m}\frac{1}{\left(1-v^2/c^2\right)^{3/2}} \tag{34}$$

$$\frac{qE}{m}\,dt = \frac{du}{\left(1-v^2/c^2\right)^{3/2}} \tag{35}$$

We can now integrate both sides, noting from the boundary conditions that if time runs from 0 to t, the velocity runs from 0 to v.

$$\int_0^v \frac{dv}{\left(1-v^2/c^2\right)^{3/2}} = \int_0^t \frac{qE}{m}\,dt \tag{36}$$

$$\left.\frac{v}{\sqrt{1-v^2/c^2}}\right|_0^v = \left.\frac{qEt}{m}\right|_0^t \tag{37}$$

$$\frac{v}{\sqrt{1-v^2/c^2}} = \frac{qEt}{m} \tag{38}$$

Solving for v, we first square both sides ...

$$\frac{v^2}{1-v^2/c^2} = \frac{q^2E^2t^2}{m^2} \tag{39}$$

$$v^2 = \left(1-\frac{v^2}{c^2}\right)\frac{q^2E^2t^2}{m^2} \tag{40}$$

$$v^2\left(1+\frac{q^2E^2t^2}{m^2c^2}\right) = \frac{q^2E^2t^2}{m^2} \tag{41}$$

$$v = \frac{qEt/m}{\sqrt{1+(qEt/mc)^2}} \tag{42}$$

We can find position by integrating v through time from 0 to t. Since v is only a function of time above, this is straightforward.

$$x = \int_0^t \frac{qEt/m}{\sqrt{1+(qEt/mc)^2}}\,dt = \frac{mc^2}{qE}\sqrt{1+\left(\frac{qEt}{mc}\right)^2}\Bigg|_0^t = \frac{mc^2}{qE}\left(\sqrt{1+\left(\frac{qEt}{mc}\right)^2}-1\right) \tag{43}$$

Classically, we would expect a parabolic path, but in relativity we find the path is a *hyperbola*. Also note that the position, velocity, and acceleration depend overall on the ratio between the particle's rest energy mc^2 to the electric force qE (note energy/force is distance).

Double check: If we let the ratio v/c tend to zero, we should recover the classical result for acceleration:

$$a = \frac{qE}{m}\frac{1}{(1-v^2/c^2)^{3/2}} \qquad \lim_{\gamma\to 0} a = \frac{qE}{m} \tag{44}$$

As a separate check on the velocity expression, we note that if the classical acceleration is qE/m, and $v = at$ in the classical limit. Plugging that in our expression for velocity,

$$v = \frac{qEt/m}{\sqrt{1+(qEt/mc)^2}} = \frac{at}{\sqrt{1+(at/c)^2}} \tag{45}$$

If $v \ll c$, then the denominator tends toward 1, and we recover $v = at$. We can do the same for the expression for x to show that it reduces to the correct classical limit. Finally, dimensional analysis will show that the units work out correctly for all three expressions - the ratio qE/mc has units of force/momentum which gives inverse seconds, so qEt/mc is dimensionless. The ratio qE/m is force/mass which has units of acceleration, so qEt/m has units of velocity. The ratio mc^2/qE has units of energy/force which gives distance.

6. Show that the angular frequency of a charge moving in a uniform magnetic field B is given by

$$\omega = \frac{qB}{m}\sqrt{1 - u^2/c^2}$$

Solution: If we have a charge moving with "angular frequency" in a uniform magnetic field, we should immediately recognize that we have uniform circular motion, which implies the charge's velocity is perpendicular to the magnetic field.

Find/given: We wish to find the angular frequency of a charge q of mass m moving at velocity u given a magnetic field B.

Sketch:

Relevant equations: We need the relativistic equation for force, the constraint for circular motion, the magnetic force, the definition of angular velocity, and the definition of γ (given in previous problems). Note that the constraint on acceleration for circular motion is a purely geometric result, and holds true with or without relativity. Since v and B are at a right angle, the magnetic force may be written as simply qvB.

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = \gamma m \frac{dv}{dt} + m v \frac{d\gamma}{dt} \tag{46}$$

$$\frac{v^2}{r} = \left(\frac{dv}{dt}\right) \text{circ} \tag{47}$$

$$F_B = qvB \tag{48}$$

$$\omega = \frac{v}{r} \tag{49}$$

Symbolic solution: We set the total force equal to the magnetic force. In uniform circular motion, speed is constant, but since the direction is constantly changing velocity is not constant. However, γ depends only on v^2, i.e., on the speed, so $d\gamma/dt$ is zero. Thus,

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = \gamma m \frac{dv}{dt} + m v \frac{d\gamma}{dt} = \gamma m \frac{dv}{dt} \tag{50}$$

Using the circular motion constraint,

$$F = qvB = \gamma m \frac{dv}{dt} = \gamma m \frac{v^2}{r} = \gamma m v \omega \tag{51}$$

$$\omega = \frac{qvB}{m\gamma v} = \frac{qB}{m\gamma} = \frac{qB}{m}\sqrt{1 - \frac{v^2}{c^2}} \tag{52}$$

Numeric solution: n/a

Double check: Dimensionally, our answer is correct: v^2/c^2 is dimensionless, and qB/m has units of inverse seconds as required. In the limit $\frac{v}{c} \ll 1$, the expression under the radical is ≈ 1, and we recover the familiar classical result.

7. The "effective mass" of a photon (bundle of electromagnetic radiation of zero rest mass and energy hf) can be determined from the relationship $m = E/c^2$. Compute the "effective mass" for a photon of wavelength 500 nm (visible) and for a photon of wavelength 0.1 nm (X-ray).

Solution: This problem is looking ahead to our discussion of radiation and energy quanta coming up shortly.

Find/given: Given the wavelength of a photon, and thus its energy, we are to find the rest mass that has the same energy.

Relevant equations: We need the photon energy, the wavelength-frequency relationship, and the rest mass-energy relationship:

$$E = hf \tag{53}$$

$$\lambda f = c \tag{54}$$

$$m = \frac{E}{c^2} \tag{55}$$

Symbolic solution: Putting the photon energy in terms of wavelength, we have

$$E = hf = h\left(\frac{c}{\lambda}\right) = \frac{hc}{\lambda} \tag{56}$$

We now wish to equate this energy to the rest energy of a particle with mass:

$$\frac{hc}{\lambda} = mc^2 \tag{57}$$

$$m = \frac{h}{\lambda c} \tag{58}$$

Numeric solution: Using $h \approx 6.626 \times 10^{-34}\,\mathrm{J \cdot s}$, $c \approx 3 \times 10^8\,\mathrm{m/s}$, and the given wavelengths, one finds

$$m = 4.4 \times 10^{-36}\,\mathrm{kg} \qquad \text{for} \qquad \lambda = 500\,\mathrm{nm} \tag{59}$$

$$m = 2.2 \times 10^{-36}\,\mathrm{kg} \qquad \text{for} \qquad \lambda = 0.1\,\mathrm{nm} \tag{60}$$

Double check: We can check that the units come out correctly: Planck's constant has units of J·s, so

$$m = \frac{[\text{J}\cdot\text{s}]}{[\text{m}][\text{m/s}]} = \frac{[\text{kg}\cdot\text{m}^2/\text{s}]}{[\text{m}^2/\text{s}]} = [\text{kg}] \tag{61}$$

8. A meterstick makes an angle of 30° with respect to the x'-axis of O'. What must be the value of v if the meterstick makes an angle of 45° with respect to the x-axis of O?

Solution: This problem involves length contraction along the x direction, but not along the y direction, so we have to handle the projections of the meter stick along each axis separately.

Find/given: Given the angle a meter stick makes with the x in one coordinate system a second coordinate system, we are to find the relative velocity of the second coordinate system.

Relevant equations: We need only the formula for length contraction along the direction of motion, $L'_x = L_x/\gamma$ and trigonometry.

Sketch:

Symbolic solution: In the rest frame of the meterstick (unprimed), we have

$$\tan\theta = \frac{L_y}{L_x} \tag{62}$$

In the frame moving with respect to the meter stick (primed), we have

$$\tan\theta' = \frac{L'_y}{L'_x} \tag{63}$$

The meter stick's length along the x direction in the primed frame will be contracted, $L'_x = L_x/\gamma$, whereas the length along the y direction will remain the same. Thus,

$$\tan\theta' = \frac{L'_y}{L'_x} = \frac{L_y}{L_x/\gamma} = \gamma\frac{L_y}{L_x} = \gamma\tan\theta \tag{64}$$

$$\gamma = \frac{\tan\theta'}{\tan\theta} = \frac{1}{\sqrt{1-v^2/c^2}} \tag{65}$$

Numeric solution:

Plugging in $\theta = 30°$ and $\theta' = 45°$, we find $\gamma = \sqrt{3}$, which gives $v = \sqrt{2/3} \approx 0.816c$.

Double check: The classical limit corresponds to $\gamma = 1$, which would imply that the angle is the same in both frames.

9. A particle appears to move with speed u at an angle θ with respect to the x axis in a certain system. At what speed and angle will this particle appear to move in a second system moving with speed v with respect to the first? Why does the answer differ from that of the previous problem?

Solution: It is most straightforward to assume that the two systems have their horizontal x axes aligned. This is still quite general, since we are still letting the particle move at an arbitrary angle θ, we may consider it to be a choice of axes and nothing more. Let the first frame, in which the particle moves with speed u at an angle θ be the 'unprimed' frame (x, y), and the second the 'primed' frame (x', y').

Along the x' direction in the primed frame, both perceived time and distance will be altered. Taking only the x' component of the velocity, we consider the particle's motion purely along the direction of relative motion of the two frames, and we may simply use our velocity addition formula. The x component of the particle's velocity will in the primed frame become

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{66}$$

Along the y' direction in the primed frame, since we consider motion of the particle orthogonal to the direction of relative motion of the frames, there is no length contraction. We need only consider time dilation. We derived this case in class, and the proper velocity addition for directions orthogonal to the relative motion leads to

$$u'_y = \frac{u_y}{\gamma\left(1 - u_x v/c^2\right)} \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{67}$$

The particle's speed in the primed frame is then easily calculated:

$$u' = \sqrt{u'_x u'_x + u'_y u'_y} = \left(\frac{u_x - v}{1 - u_x v/c^2}\right)^2 + \left(\frac{u_y}{\gamma\left(1 - u_x v/c^2\right)}\right)^2 \tag{68}$$

$$= \sqrt{\frac{\left(u_x - v\right)^2 + u_y^2/\gamma^2}{\left(1 - u_x v/c^2\right)^2}} = \frac{\sqrt{\left(u_x - v\right)^2 + u_y^2/\gamma^2}}{1 - u_x v/c^2} \tag{69}$$

As a double-check, we can set $\theta = 0$, such that $u_y = 0$, which corresponds to the particle moving along the x axis. Our expression then reduces to the usual one-dimensional velocity addition formula.

The direction of motion in the primed frame is also found readily:

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{u_y}{\gamma\left(1 - u_x v/c^2\right)} \frac{1 - u_x v/c^2}{u_x - v} = \left(\frac{u_y}{u_x - v}\right)\sqrt{1 - v^2/c^2} \tag{70}$$

10. Consider a charged parallel-plate capacitor, whose electric field (in its rest frame) is uniform (neglecting edge effects) between the plates and zero outside. Find the electric field according to an observer in motion at constant velocity v **(a)** along a line running through the center of the capacitor between the plates, and **(b)** along a line perpendicular to the plates.

Solution: See the lecture notes on radiation, `http://faculty.mint.ua.edu/~pleclair/PH253/Notes/blackbody.pdf`, where I've worked this problem out in detail.