UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 253 / LeClair Fall 2010

Problem Set 2: Radiation

Instructions:

1. Answer all questions below. Show your work for full credit.
2. All problems are due Fri 3 September 2010 by the end of the day.
3. You may collaborate, but everyone must turn in their own work.

1. In a hydrogen atom an electron of charge $-e$ orbits around a proton of charge $+e$.

(a) Find the total energy E and the orbital frequency ω as a function of r, the distance between the electron and proton.
(b) Calculate the energy radiated per unit time as a function of r.
(c) Using $dr/dt = (dr/dE)(dE/dt)$, find the time it takes for a hydrogen atom to collapse from a radius of 10^{-9}m to a radius of 0.

2. Assume the sun radiates like a black body at 5500 K. Assume the moon absorbs all the radiation it receives from the sun and reradiates an equal amount of energy like a black body at temperature T. The angular diameter of the sun seen from the moon is about 0.01 rad. What is the equilibrium temperature T of the moon's surface? (Note: you do not need any other data than what is contained in the statement above.

3. The time average of some function $f(t)$ taken over an interval T is given by

$$\langle f(t) \rangle = \frac{1}{T} \int_{t}^{T+t} f(t')\, dt' \tag{1}$$

where t' is just a dummy variable of integration. If $\tau = 2\pi/\omega$ is the period of a harmonic function, show that

$$\langle \sin^2(kx - \omega t) \rangle = \frac{1}{2} \tag{2}$$

$$\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2} \tag{3}$$

$$\langle \sin(kx - \omega t) \cos(kx - \omega t) \rangle = 0 \tag{4}$$

when $T = \tau$ and when $T \gg \tau$.

4. As a function of wavelength, Planck's law states that the emitted power of a black body per unit area of emitting surface, per unit wavelength is

$$I(\lambda, T) = \frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} \tag{5}$$

That is, $I(\lambda, T)d\lambda$ gives the emitted power per unit area emitted between wavelengths λ and $\lambda + d\lambda$. Show by differentiation that the wavelength λ_m at which $I(\lambda, T)$ is maximum satisfies the relationship

$$\lambda_m T = b \tag{6}$$

where b is a constant. This result is known as *Wien's Displacement Law,* and can be used to determine the temperature of a black body radiator from only the peak emission wavelength. The constant above has a numerical value of $b = 2.9 \times 10^6$ nm-K. *Note: at some point you will need to solve an equation numerically.*

5. Presume the surface temperature of the sun to be 5500 K, and that it radiates approximately as a blackbody. What fraction of the sun's energy is radiated in the visible range of $\lambda = 400 - 700$ nm? One valid solution is to plot the energy density on graph paper and find the result numerically.

6. The equation for a driven damped oscillator is

$$\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{q}{m}E(t) \tag{7}$$

(a) Explain the significance of each term.
(b) Let $E = E_o e^{i\omega t}$ and $x = x_o e^{i(\omega t - \alpha)}$ where E_o and x_o are real quantities. Substitute into the above expression and show that

$$x_o = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \tag{8}$$

(c) Derive an expression for the phase lag α, and sketch it as a function of ω, indicating ω_o on the sketch.

7. In class, we will show that an oscillating charge of natural frequency ω_o feels a damping force due to the radiation it is emitting, governed by a damping constant γ. If the charge is driven by an external electric field oscillating sinusoidally at ω, $E(t) = E_o \cos\omega t$, we arrive at the following equation of motion for the charge:

$$x(t) = A\cos(\omega t + \varphi) \tag{9}$$

$$A = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \tag{10}$$

$$\tan\varphi = \frac{2\omega\omega_o\gamma}{\omega^2 - \omega_o^2} \tag{11}$$

In one sense, our oscillating charge looks like a dipole, which means that a system of oscillating charges looks a bit like a dielectric. One can show that a collection of N such charges per unit volume oscillating together (e.g., a dilute gas) gives the medium a dielectric constant

$$\epsilon = \epsilon_o + \frac{exN}{E} \tag{12}$$

(a) Using the expressions for $x(t)$ and $E(t)$, show that for small damping (and thus small φ) the dielectric constant can be written[i]

$$\frac{\epsilon}{\epsilon_o} = 1 + \frac{e^2N}{\epsilon_o m}\frac{\omega^2 - \omega_o^2}{(\omega^2 - \omega_o^2)^2 + (2\gamma\omega\omega_o)^2} \tag{14}$$

(b) Knowledge of the dielectric constant of a medium gives us the index of refraction as well, $n^2 = \epsilon/\epsilon_o$. Show that at low density with negligible damping ($\gamma \approx 0$) the index of refraction is approximately

$$n \approx 1 + \frac{e^2N}{2\epsilon_o m(\omega^2 - \omega_o^2)^2} \tag{15}$$

Note $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ when $x \ll 1$.

(c) In air, the natural frequency of the oscillators ω_o is in the ultraviolet, so visible light driving the oscillators has frequencies $\omega < \omega_o$. Sketch n for $\omega < \omega_o$. Will red or blue light be refracted more?

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[i] Note that for small φ,

$$\frac{\cos(\omega t + \varphi)}{\cos\omega t} \approx \cos\varphi = \cos\left[\tan^{-1}\left(\frac{2\omega\omega_o\gamma}{\omega^2 - \omega_o^2}\right)\right] = \frac{\omega^2 - \omega_o^2}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \tag{13}$$