University of Alabama
Department of Physics and Astronomy

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Problem Set 2: Solution

1. In a hydrogen atom an electron of charge $-e$ orbits around a proton of charge $+e$.

(a) Find the total energy E and the orbital frequency ω as a function of r, the distance between the electron and proton.
(b) Calculate the energy radiated per unit time as a function of r.
(c) Using $dr/dt = (dr/dE)(dE/dt)$, find the time it takes for a hydrogen atom to collapse from a radius of 10^{-9}m to a radius of 0.

Solution: The total energy is kinetic plus potential. The potential energy is that of two point charges e and $-e$ separated by a distance r. If we take the frame of reference that the (much heavier) proton is at rest, the kinetic energy is just that of the electron, to which we will assign mass m and velocity v:

$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_o r} \tag{1}$$

This equation has the electron velocity present, and we wish to find the energy as a function of radius only. We can eliminate the velocity by noting that the electric force between the proton and electron is constrained to equal the centripetal force required to maintain circular motion. That is,

$$\frac{-e^2}{4\pi\epsilon_o r^2} = -\frac{mv^2}{r} \qquad \Longrightarrow \qquad mv^2 = \frac{e^2}{4\pi\epsilon_o r} \tag{2}$$

Substituting into our first equation,

$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_o r} = \frac{e^2}{8\pi\epsilon_o r} - \frac{e^2}{4\pi\epsilon_o r} = -\frac{e^2}{8\pi\epsilon_o r} \tag{3}$$

Just like gravitational orbits, the total energy is half of the potential energy. The angular frequency is found from $v = r\omega$, or $\omega = v/r$. From our force balance above, we have v in terms of r, so

$$\omega = \frac{v}{r} = \frac{1}{r}\sqrt{\frac{e^2}{4\pi\epsilon_o mr}} = \sqrt{\frac{e^2}{4\pi\epsilon_o mr^3}} \tag{4}$$

Given that the electron is in circular motion, it is accelerating, which means it must be radiating. The Larmor formula gives us the average radiated power, or energy per unit time:

$$\frac{dE}{dt} = -\frac{e^2 a^2}{6\pi\epsilon_o c^3} \tag{5}$$

Here we have inserted the minus sign because we know that the electron is losing energy by radiating. The acceleration a can be found from our force balance above, diving through by mass m:

$$a = -\frac{v^2}{r} = -\frac{e^2}{4\pi\epsilon_o m r^2} \tag{6}$$

Using the right-most form, we can find the power in terms of radius alone:

$$\frac{dE}{dt} = -\frac{e^2 a^2}{6\pi\epsilon_o c^3} = -\frac{e^2}{6\pi\epsilon_o c^3}\left(\frac{e^2}{4\pi\epsilon_o m r^2}\right)^2 = -\frac{e^6}{96\pi^3\epsilon_o^3 m^2 c^3 r^4} \tag{7}$$

If the electron is radiating, it is losing energy, which means its orbit must be decaying. With the power in hand, we can calculate the rate at which the radius of the electron's orbit decays and figure out how long such an atom would be stable. Using the chain rule

$$\frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt} = \frac{dE}{dt}\bigg/\frac{dE}{dr} \tag{8}$$

Since dE/dt is the power we just found, we need only dE/dr:

$$\frac{dE}{dr} = \frac{d}{dr}\left(-\frac{e^2}{8\pi\epsilon_o r}\right) = \frac{e^2}{8\pi\epsilon_o r^2} \tag{9}$$

Putting it together,

$$\frac{dr}{dt} = \frac{dE}{dt}\bigg/\frac{dE}{dr} = -\frac{e^6}{96\pi^3\epsilon_o^3 m^2 c^3 r^4}\left(\frac{8\pi\epsilon_o r^2}{e^2}\right) = -\frac{e^4}{12\pi^2\epsilon_o^2 m^2 c^3 r^2} = -\left(\frac{e^4}{12\pi^2\epsilon_o^2 m^2 c^3}\right)\frac{1}{r^2} \tag{10}$$

For convenience, let $C = \frac{e^4}{12\pi^2\epsilon_o^2 m^2 c^3}$. This hideous combination is just a constant anyway, lumping it all together means we just have to keep track of one constant instead of 6. Our equation then reads

$$\frac{dr}{dt} = -\frac{C}{r^2} \tag{11}$$

This equation is separable[i]:

$$r^2\,dr = -C\,dt \tag{12}$$

Integrating both sides, and noting that we start at time $t=0$ at radius $r_i = 10^{-9}$ m and end at time t with radius zero,

$$\int_{r_i}^{0} r^2\,dr = -\frac{1}{3} r_i^3 = \int_{0}^{t} -C\,dt = -Ct \tag{13}$$

$$t = \frac{r_i^3}{3C} \tag{14}$$

Substituting our definition of C, the time for the electron to reach the proton is

$$t = \frac{4\pi^2 \epsilon_o^2 m^2 c^3}{e^4} r_i^3 \tag{15}$$

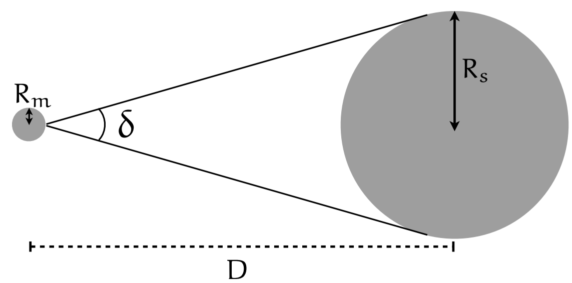
With the given radius of $r_i = 10^{-9}$ m, $t \sim 10^{-7}$ s. Using a more realistic radius for the lowest energy state of a hydrogen atom, $r_i \approx 5 \times 10^{-11}$ m, one finds $t \sim 10^{-11}$ s.

Moral of the story: classical atoms are not stable.

[i] If we close our eyes and manipulate the differentials like fractions, we would cross multiply to separate the equation.

2. Assume the sun radiates like a black body at 5500 K. Assume the moon absorbs all the radiation it receives from the sun and reradiates an equal amount of energy like a black body at temperature T. The angular diameter of the sun seen from the moon is about 0.01 rad. What is the equilibrium temperature T of the moon's surface? (Note: you do not need any other data than what is contained in the statement above.

Solution: The geometry of the problem is shown below, where δ is the angular diameter, R_m the moon's radius, R_s the sun's radius, and D the sun-moon distance.



The *definition* of angular diameter[ii], using the distances in the figure above, is

$$\tan\frac{\delta}{2} = \frac{R_s}{D} \tag{16}$$

With geometry in hand, we now need to balance the sun's power received by the moon with the power that the moon will re-radiate by virtue of its being at temperature T_m. Any body at temperature T emits a power $P =! \sigma T^4 A$, where A is the area over which the radiation is emitted and σ is a constant. Thus, since the sun emits radiation over its whole surface area $4\pi R_s^2$,

$$P_s = \sigma T_s^4 \left(4\pi R_s^2\right) \tag{17}$$

At a distance D corresponding to the moon's position, this power is spread over a sphere of radius D and surface area $4\pi D^2$. The amount of power the moon receives just depends on the ratio its absorbing area to the total area over which the power is spread out. The moon absorbs radiation over an area corresponding to its cross section, πR_m^2, so the fraction of the sun's total power that

[ii] See, e.g., `http://en.wikipedia.org/wiki/Angular_diameter`

the moon receives is $\pi R_m^2/4\pi D^2$. Thus, the moon receives a power

$$P_{mr} = P_s \frac{\pi R_m^2}{4\pi D^2} = P_s \frac{R_m^2}{4D^2} = \sigma T_s^4 \left(4\pi R_s^2\right) \frac{R_m^2}{4D^2} \tag{18}$$

Absorbing this radiation from the sun will cause the moon to heat up to temperature T_m, and it will re-emit radiation as a black body at temperature T_m. Though the moon absorbs over its cross-sectional area, it emits over its whole surface area, so its emitted power is

$$P_{me} = \sigma T_m^4 \left(4\pi R_m^2\right) \tag{19}$$

Equilibrium requires that the power the moon receives equal the power the moon emits, so

$$P_{mr} = P_{me} \tag{20}$$

$$\sigma T_s^4 \left(4\pi R_s^2\right) \frac{R_m^2}{4D^2} = \sigma T_m^4 \left(4\pi R_m^2\right) \tag{21}$$

$$T_s^4 \frac{R_s^2}{4D^2} = T_m^4 \tag{22}$$

$$T_m = T_s \sqrt{\frac{R_s}{2D}} = T_s \sqrt{\frac{1}{2} \tan \frac{\delta}{2}} \approx 275\,\text{K} \tag{23}$$

Compare this with a mean lunar surface temperature at the equator of 220 K – not bad given the approximate geometry, and complete ignorance of reflection! It is interesting to see that the moon's radius does not factor in at all – it determines both the absorbed and emitted power in exactly the same way, and ends up canceling out.

3. The time average of some function $f(t)$ taken over an interval T is given by

$$\langle f(t) \rangle = \frac{1}{T} \int\limits_{t}^{T+t} f(t')\, dt' \tag{24}$$

where t' is just a dummy variable of integration. If $\tau = 2\pi/\omega$ is the period of a harmonic function, show that

$$\langle \sin^2(kx - \omega t) \rangle = \frac{1}{2} \tag{25}$$

$$\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2} \tag{26}$$

$$\langle \sin(kx - \omega t) \cos(kx - \omega t) \rangle = 0 \tag{27}$$

when $T = \tau$ and when $T \gg \tau$.

Solution: Starting with $\langle \sin^2(kx - \omega t)\rangle$, is is convenient to use a substitution

$$u = kx - \omega t' \tag{28}$$
$$du = -w\, dt' \tag{29}$$

Then we have

$$\begin{aligned}
\langle f(t)\rangle = \langle \sin^2(kx-\omega t)\rangle &= \frac{1}{T}\int\limits_{t}^{T+t} \sin^2\left(kx-\omega t'\right) dt' = \frac{-1}{\omega T}\int\limits_{t'=t}^{t'=T+t} \sin^2 u\, du \\
&= \frac{-1}{2\omega T}\left[u - \sin u \cos u\right]\Bigg|_{t'=t}^{t'=T+t} = \frac{-1}{2\omega T}\left[(kx-\omega t) - \sin(kx-\omega t)\cos(kx-\omega t)\right]\Bigg|_{t}^{T+t} \\
&= \frac{-1}{2\omega T}\left[-\omega T - \sin(kx-\omega t-\omega T)\cos(kx-\omega t-\omega T) + \sin(kx-\omega t)\cos(kx-\omega t)\right] \\
&= \frac{1}{2} + \frac{1}{2\omega T}\Big(\sin(kx-\omega t-\omega T)\cos(kx-\omega t-\omega T) - \sin(kx-\omega t)\cos(kx-\omega t)\Big)
\end{aligned} \tag{30}$$

For the specific limit of $T = \tau = \frac{2\pi}{\omega}$, we note that $\omega T = \omega\tau = 2\pi$. Thus,

$$\langle \sin^2(kx-\omega t)\rangle = \frac{1}{2} + \frac{1}{2\omega T}\Big(\sin(kx-\omega t-2\pi)\cos(kx-\omega t-2\pi) - \sin(kx-\omega t)\cos(kx-\omega t)\Big) \tag{31}$$

Since $\sin(\theta \pm 2\pi) = \sin\theta$ and $\cos(\theta \pm 2\pi) = \cos\theta$, the second term vanishes, we have

$$\langle \sin^2(kx-\omega t)\rangle = \frac{1}{2} \tag{32}$$

In the limit $T \gg \tau$, we notice that the second term in Eq. 30 goes as T in the denominator, while the sin and cos functions in the numerator are each at most 1. Thus, the entire second term goes as (const)/T. In the limit of large T (say $T \to \infty$), this term vanishes.

$$\lim_{T\to\infty}\langle f(t)\rangle = \lim_{T\to\infty}\frac{1}{2} + \frac{\sin(kx-\omega t-2\pi)\cos(kx-\omega t-2\pi) - \sin(kx-\omega t)\cos(kx-\omega t)}{2\omega T} = \frac{1}{2}$$

For the second part, all we need to do is notice that

$$\int \sin^2 u\, du = \frac{1}{2}u - \frac{1}{2}\sin u \cos u + C \tag{33}$$
$$\int \cos^2 u\, du = \frac{1}{2}u + \frac{1}{2}\sin u \cos u + C \tag{34}$$

Both integrals are the same, except for the change of sign of the second term. In both limits considered, the second term vanishes, so its sign is irrelevant.[iii] Thus,

$$\langle \sin^2(kx-\omega t)\rangle = \langle \cos^2(kx-\omega t)\rangle = \frac{1}{2} \qquad T \gg \tau, T=\tau \tag{35}$$

Finally, we are left with

$$\langle f(t)\rangle = \langle \sin(kx-\omega t)\cos(kx-\omega t)\rangle \tag{36}$$

Using the same substitution above, we find

$$\begin{aligned}\langle f(t)\rangle = \langle \sin(kx-\omega t)\cos(kx-\omega t)\rangle &= \frac{1}{T}\int\limits_{t}^{T+t} \sin(kx-\omega t)\cos(kx-\omega t)\,dt' \\ &= \frac{-1}{\omega T}\int\limits_{t'=t}^{t'=T+t} \sin u \cos u\, du = \frac{-1}{\omega T}\left(\frac{1}{2}\sin^2 u\right)\Bigg|_{t'=t}^{t'=T+t} = \frac{-1}{4\omega T}\left(1-\cos 2u\right)\Bigg|_{t'=t}^{t'=T+t} \\ &= \frac{-1}{4\omega T}\left(-\cos(2kx-2\omega t-2\omega T)+\cos(2kx-\omega t)\right)\end{aligned} \tag{37}$$

At the limit $T=\tau$, since $\omega T=\omega\tau=2\pi$ the two terms in brackets cancel since $\cos\theta=\cos(\theta\pm 2\pi)$. In the limit $T\gg\tau$, we note

$$\lim_{T\to\infty}\frac{-\cos(2kx-2\omega t-2\omega T)+\cos(2kx-\omega t)}{4\omega T}=0 \tag{38}$$

since the numerator can be at most 2 for any value of T. Thus,

$$\langle \sin(kx-\omega t)\cos(kx-\omega t)\rangle = 0 \qquad T\gg\tau, T=\tau \tag{39}$$

4. As a function of wavelength, Planck's law states that the emitted power of a black body per unit area of emitting surface, per unit wavelength is

$$I(\lambda,T)=\frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}}-1\right]^{-1} \tag{40}$$

That is, $I(\lambda,T)d\lambda$ gives the emitted power per unit area emitted between wavelengths λ and $\lambda + d\lambda$. Show by differentiation that the wavelength λ_m at which $I(\lambda,T)$ is maximum satisfies the

[iii] If you like, repeat everything above with the appropriate signs flipped.

relationship

$$\lambda_m T = b \tag{41}$$

where b is a constant. This result is known as *Wien's Displacement Law,* and can be used to determine the temperature of a black body radiator from only the peak emission wavelength. The constant above has a numerical value of $b = 2.9 \times 10^6$ nm-K. *Note: at some point you will need to solve an equation numerically.*

Solution: First, we must find $dI/d\lambda$. Strictly, we want $\partial I/\partial\lambda$, since we are presuming constant temperature, but that is only a formal point since T does not depend on λ. For convenience, define the following substitutions:

$$a \equiv 8\pi hc^2 \tag{42}$$

$$b \equiv \frac{hc}{kT} \tag{43}$$

Thus,

$$I(\lambda, T) = \frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} = \frac{a}{\lambda^5}\left[e^{\frac{b}{\lambda}} - 1\right]^{-1} \tag{44}$$

$$\frac{dI}{d\lambda} = \frac{-5a}{\lambda^6}\frac{1}{e^{\frac{b}{\lambda}} - 1} + \frac{-a}{\lambda^5}\left(\frac{1}{e^{\frac{b}{\lambda}} - 1}\right)^2\left(\frac{-be^{\frac{b}{\lambda}}}{\lambda^2}\right) = \left(\frac{a}{\lambda^7}\right)\frac{be^{\frac{b}{\lambda}} - 5\lambda e^{\frac{b}{\lambda}} + 5\lambda}{\left(e^{\frac{b}{\lambda}} - 1\right)^2} = 0 \tag{45}$$

Finding the maximum of $I(\lambda, T)$ with respect to λ means setting $dI(\lambda, T)/d\lambda = 0$.[iv] The denominator in the equation above is then irrelevant, as is the λ^{-7} prefactor, and we have

$$0 = be^{\frac{b}{\lambda}} - 5\lambda e^{\frac{b}{\lambda}} + 5\lambda \tag{46}$$

$$0 = be^{\frac{b}{\lambda}} + 5\lambda\left(1 - e^{\frac{b}{\lambda}}\right) \tag{47}$$

$$5 = \frac{be^{\frac{b}{\lambda}}}{\lambda\left(e^{\frac{b}{\lambda}} - 1\right)} \tag{48}$$

We can make another substitution to make things easier. Define $x \equiv \frac{b}{\lambda} = \frac{hc}{\lambda kT}$ and simplify:

$$\frac{xe^x}{e^x - 1} - 5 = 0 \tag{49}$$

[iv]Since we know the curve is concave downward, we won't bother with the second derivative test; we know very well we will find a maximum and not a minimum.

If we find the root of this equation, we have (after undoing our substitutions) the value of λ for which $I(\lambda, T)$ is maximum. Unfortunately, there is no analytic solution. Using Newton's method or something similar, we find the root is

$$x = \frac{hc}{\lambda kT} \approx 4.695 \tag{50}$$

Solving for λ, we obtain the desired result:

$$\lambda_{\text{max}} \approx \frac{hc}{4.965kT} \approx \frac{2.898 \times 10^6 \,\text{nm} \cdot \text{K}}{T} \tag{51}$$

5. Presume the surface temperature of the sun to be 5500 K, and that it radiates approximately as a blackbody. What fraction of the sun's energy is radiated in the visible range of $\lambda = 400 - 700\,\text{nm}$? One valid solution is to plot the energy density on graph paper and find the result numerically.

Solution: The emitted power per unit area per unit wavelength for a blackbody is given in a previous problem:

$$I(\lambda, T) = \frac{8\pi hc^2}{\lambda^5} \left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} \tag{52}$$

The power per unit area emitted over a range of wavelengths λ_1 to λ_2 is found by integrating $I(\lambda, T)$ over those limits, and the total power is integrating over all wavelengths from 0 to ∞. The fraction we desire is then the power over wavelengths λ_1 to λ_2 divided by the total power:

$$f = (\text{fraction}) = \frac{\int\limits_{\lambda_1}^{\lambda_2} I(\lambda, T)\, d\lambda}{\int\limits_0^{\infty} I(\lambda, T)\, d\lambda} \tag{53}$$

Let us first worry about the indefinite integral and put it in a bit simpler form.

$$\int I(\lambda, T)\, d\lambda = \int \frac{8\pi hc^2}{\lambda^5} \left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} d\lambda \tag{54}$$

It is convenient to make a change of variables to

$$u = \frac{hc}{\lambda k_b T} \qquad \text{or} \qquad \lambda = \frac{hc}{u k_b T} \tag{55}$$

This substitution implies

$$du = \frac{hc}{k_bT}\left(\frac{-d\lambda}{\lambda^2}\right) = -\frac{hc}{k_bT}\left(\frac{k_bTu}{hc}\right)^2 d\lambda = -\frac{u^2k_bT}{hc}d\lambda \tag{56}$$

$$d\lambda = -\frac{hc}{u^2k_bT}du \tag{57}$$

Performing the substitution,

$$\int I(\lambda, T)\, d\lambda = \int \frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_bT}} - 1\right]^{-1} d\lambda = \int \frac{8\pi hc^2u^5k_b^5T^5}{h^5c^5}\frac{1}{e^u - 1}\frac{-hc}{u^2k_bT}du \tag{58}$$

$$= -\frac{8\pi k_b^4T^4}{h^3c^2}\int \frac{u^3}{e^u - 1}\, du \tag{59}$$

The overall constants multiplying the integral will cancel in the fraction we wish to find:

$$f = \frac{\frac{8\pi k_b^4T^4}{h^3c^2}\int\limits_{u_1}^{u_2}\frac{u^3}{e^u-1}\, du}{\frac{8\pi k_b^4T^4}{h^3c^2}\int\limits_{0}^{\infty}\frac{u^3}{e^u-1}\, du} = \frac{\int\limits_{u_1}^{u_2}\frac{u^3}{e^u-1}\, du}{\int\limits_{\infty}^{0}\frac{u^3}{e^u-1}\, du} \tag{60}$$

Here the new limits of integration for the numerator are $u_1 = \frac{hc}{\lambda_1 k_b T} \approx 6.55\,\mathrm{m}^{-1}$ and $u_2 = \frac{hc}{\lambda_1 k_b T} \approx 3.74\,\mathrm{m}^{-1}$, and the denominator has limits of ∞ and 0 after the substitution.

$$f = \frac{\int\limits_{6.55}^{3.74}\frac{u^3}{e^u-1}\, du}{\int\limits_{\infty}^{0}\frac{u^3}{e^u-1}\, du} \tag{61}$$

As it turns out, the integral in the denominator is known, and has a numerical value of $\pi^4/15$. (At the end of the solutions set, we show how this may be deduced.) The integral in the numerator has no closed-form solution, and must be found numerically. One thing we notice is that the denominator contains a factor $e^u - 1$, and at the limits of integration we have

$$e^{3.74} \approx 42 \tag{62}$$

$$e^{6.55} \approx 700 \tag{63}$$

In this case, since $e^u \gg 1$, to a good approximation we can write

$$\frac{1}{e^u - 1} \approx \frac{1}{e^u} = e^{-u} \tag{64}$$

The error we make in this approximation is in the worst case of order $1/43 \sim 2\%$ This makes the integral in the numerator of our fraction a known one, which can be integrated by parts[v]:

$$\int_{6.55}^{3.74} \frac{u^3}{e^u - 1}\,du \approx \int_{6.55}^{3.74} \frac{u^3}{e}{}^{-u}\,du = e^{-u}\left(u^3 + 3u^2 + 6u + 6\right)\Big|_{6.55}^{3.74} \approx 2.29 \tag{65}$$

Thus,

$$f \approx \frac{2.29}{\pi^4/15} \approx 0.35 \tag{66}$$

About 35% of the sun's radiation should be in the visible range.[vi] A more exact numerical calculation gives closer to 36%, meaning our approximation above was indeed accurate to about 2%.

6. The equation for a driven damped oscillator is

$$\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{q}{m}E(t) \tag{67}$$

(a) Explain the significance of each term.
(b) Let $E = E_o e^{i\omega t}$ and $x = x_o e^{i(\omega t - \alpha)}$ where E_o and x_o are real quantities. Substitute into the above expression and show that

$$x_o = \frac{qE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \tag{68}$$

(c) Derive an expression for the phase lag α, and sketch it as a function of ω, indicating ω_o on the sketch.

Solution: The significance of each term is probably more apparent if we re-arrange and multiply by mass:

$$m\frac{d^2x}{dt^2} = -m\omega_o^2 x - 2\gamma m\omega_o\frac{dx}{dt} + qE(t) \tag{69}$$

The term on the right is the net force on the oscillator. The first term on the left is the restoring force, the second the viscous damping term, and the last the driving force of the oscillator.

[v] Or with Wolfram ...

[vi] This is what *leaves the sun*, to figure out what reaches the earth's surface we would have to account for reflection and absorption by the atmosphere. The fraction of visible light is closer to 42% at the earth's surface; see `uvb.nrel.colostate.edu/UVB/publications/uvb_primer.pdf` for example.

First, we find the derivatives of x, noting $i^2 = -1$:

$$\frac{dx}{dt} = i\omega x_o e^{i(\omega t - \alpha)} \tag{70}$$
$$\frac{d^2x}{dt^2} = -\omega^2 x_o e^{i(\omega t - \alpha)} \tag{71}$$

Substituting into the original equaiton,

$$\frac{q}{m} E_o e^{i\omega t} = -\omega^2 x_o e^{i(\omega t - \alpha)} + 2\gamma\omega_o i\omega x_o e^{i(\omega t - \alpha)} + \omega_o^2 x_o e^{i(\omega t - \alpha)} \tag{72}$$
$$\frac{q}{m} E_o e^{i\omega t} = e^{i(\omega t - \alpha)} \left(-\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o\right) \tag{73}$$
$$\frac{q}{m} E_o e^{i\omega t} = e^{i\omega t} e^{-i\alpha} \left(-\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o\right) \tag{74}$$
$$\frac{qE_o}{m} e^{i\alpha} = -\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o \tag{75}$$

To proceed, we use the Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{76}$$

Giving

$$\frac{qE_o}{m} (\cos\alpha + i\sin\alpha) = -\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o \tag{77}$$

We now have two separate equations if we equate the purely real and purely imaginary parts:

$$\frac{qE_o}{m} \cos\alpha = \omega_o^2 x_o - \omega^2 x_o \tag{78}$$
$$\frac{qE_o}{m} \sin\alpha = 2\gamma\omega\omega_o x_o \tag{79}$$

We can square both equations and add them together:

$$\frac{q^2E_o^2}{m^2} \left(\cos^2\alpha + \sin^2\alpha\right) = \left(\omega_o^2 - \omega^2\right)^2 x_o^2 + (2\gamma\omega\omega_o)^2 x_o^2 \tag{80}$$
$$x_o^2 = \frac{q^2E_o^2}{m^2} \frac{1}{\left(\omega_o^2 - \omega^2\right)^2 x_o^2 + (2\gamma\omega\omega_o)^2} \tag{81}$$
$$x_o = \frac{qE_o}{m} \frac{1}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 x_o^2 + (2\gamma\omega\omega_o)^2}} \tag{82}$$

This is the desired amplitude of vibration. Going back to the preceding two equations, we can divide the second equation by the first to find the phase angle:

$$\tan\alpha = \frac{2\gamma\omega\omega_o}{\omega_o^2-\omega^2} \tag{83}$$

This is the same phase angle derived in the notes (modulo an overall sign due to the convention chosen), a sketch of phase angle versus frequency is provided there.

7. In class, we will show that an oscillating charge of natural frequency ω_o feels a damping force due to the radiation it is emitting, governed by a damping constant γ. If the charge is driven by an external electric field oscillating sinusoidally at ω, $E(t)=E_o\cos\omega t$, we arrive at the following equation of motion for the charge:

$$x(t) = A\cos(\omega t+\varphi) \tag{84}$$

$$A = \frac{eE_o/m}{\sqrt{(\omega_o^2-\omega^2)^2+(2\gamma\omega\omega_o)^2}} \tag{85}$$

$$\tan\varphi = \frac{2\omega\omega_o\gamma}{\omega^2-\omega_o^2} \tag{86}$$

In one sense, our oscillating charge looks like a dipole, which means that a system of oscillating charges looks a bit like a dielectric. One can show that a collection of N such charges per unit volume oscillating together (e.g., a dilute gas) gives the medium a dielectric constant

$$\epsilon = \epsilon_o + \frac{exN}{E} \tag{87}$$

(a) Using the expressions for $x(t)$ and $E(t)$, show that for small damping (and thus small φ) the dielectric constant can be written[vii]

$$\frac{\epsilon}{\epsilon_o} = 1 + \frac{e^2N}{\epsilon_o m}\frac{\omega^2-\omega_o^2}{(\omega^2-\omega_o^2)^2+(2\gamma\omega\omega_o)^2} \tag{89}$$

(b) Knowledge of the dielectric constant of a medium gives us the index of refraction as well, $n^2=\epsilon/\epsilon_o$. Show that at low density with negligible damping ($\gamma\approx 0$) the index of refraction is

[vii]Note that for small φ,

$$\frac{\cos(\omega t+\varphi)}{\cos\omega t} \approx \cos\varphi = \cos\left[\tan^{-1}\left(\frac{2\omega\omega_o\gamma}{\omega^2-\omega_o^2}\right)\right] = \frac{\omega^2-\omega_o^2}{\sqrt{(\omega_o^2-\omega^2)^2+(2\gamma\omega\omega_o)^2}} \tag{88}$$

approximately

$$n \approx 1 + \frac{e^2 N}{2\epsilon_o m \left(\omega^2 - \omega_o^2\right)^2} \tag{90}$$

Note $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ when $x \ll 1$.

(c) In air, the natural frequency of the oscillators ω_o is in the ultraviolet, so visible light driving the oscillators has frequencies $\omega < \omega_o$. Sketch n for $\omega < \omega_o$. Will red or blue light be refracted more?

Appendix: Evaluating $\int_0^\infty x^3\,dx/\left(e^x - 1\right)$

Pathologically, the best way to calculate the integral

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx \tag{91}$$

is to calculate a more general case and reduce it to the answer we require. Take the following integral

$$\int_0^\infty \frac{x^n}{e^x - 1}\,dx = \int_0^\infty \frac{x^n e^{-x}}{1 - e^{-x}}\,dx \tag{92}$$

The denominator is always less than one, and is in fact the sum of a geometric series with common multiplier e^{-x}:

$$\frac{1}{1 - e^{-x}} = \sum_{k=0}^{\infty} e^{-kx} \tag{93}$$

If we substitute in this series, our integral becomes

$$\int_0^\infty x^n e^{-x} \sum_{k=0}^{\infty} e^{-kx}\,dx \tag{94}$$

We can bring the factor e^{-x} inside our summation, which only shifts the lower limit of the sum from 0 to 1, leaving:

$$\int_0^\infty x^n \sum_{k=1}^{\infty} e^{-kx}\, dx \tag{95}$$

Now make a change of variables $u = kx$, meaning

$$x^n = \frac{u^n}{k^n} \tag{96}$$
$$dx = \frac{du}{k} \tag{97}$$

With this change of variables, our integral is:

$$\int_0^\infty \frac{u^n}{k^n} \sum_{k=1}^{\infty} e^{u} \frac{du}{k} = \int_0^\infty u^n \sum_{k=1}^{\infty} e^{u} \frac{du}{k^{n+1}} \tag{98}$$

Each term in the sum represents an integral over u, all of which are convergent. This means we can interchange the order of summation and integration:

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \int_0^\infty u^n e^{-u}\, du \tag{99}$$

The integral on the right side is the definition of the Gamma function $\Gamma(n+1)$, while the summation is then the definition of the Riemann zeta function $\zeta(n+1)$. Thus,

$$\int_0^\infty \frac{x^n}{e^x - 1}\, dx = \zeta(n+1)\Gamma(n+1) \tag{100}$$

With $n=3$,

$$\zeta(n+1) = \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{101}$$
$$\Gamma(n+1) = n! = 3! = 6 \tag{102}$$

And finally,

$$\int_0^\infty \frac{x^3}{e^x - 1}\, dx = \zeta(n+1)\Gamma(n+1) = \frac{\pi^4}{15} \tag{103}$$