University of Alabama
Department of Physics and Astronomy

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Problem Set 3: Solutions

1. The emitter in a photoelectric tube has a threshold wavelength of 600 nm. Determine the wavelength of the light incident on the tube if the stopping potential for this light is 2.5 V.

Solution: Recall the photoelectric equation

$$e\Delta V_{\text{stop}} = E_{\text{photon}} - W \tag{1}$$

where ΔV_{stop} is the stopping potential, E_{photon} the photon energy of the incident light, and W the binding energy (work function) of the material. The "threshold" wavelength λ_t is the maximum wavelength (and thus minimum energy or frequency) of incident light that is capable of ejecting electrons. Thus, at this threshold wavelength, the incident photon energy must be equal to the work function, so a measurement of the critical wavelength gives you the work function:

$$E_{\text{photon}} = \frac{hc}{\lambda_t} = W \qquad \text{at threshold} \tag{2}$$

If the stopping potential for a particular wavelength λ is ΔV, that means an electron of charge e moving through that potential difference acquires an energy of $e\Delta V$. An energy balance gives

$$e\Delta V = \frac{hc}{\lambda} - W = \frac{hc}{\lambda} - \frac{hc}{\lambda_t} \tag{3}$$

Solving for λ,

$$\frac{hc}{\lambda} = e\Delta V + \frac{hc}{\lambda_t} \tag{4}$$

$$\frac{1}{\lambda} = \frac{e\Delta V}{hc} + \frac{1}{\lambda_t} \tag{5}$$

Noting that $hc = 1240\,\text{eV}\cdot\text{nm}$,

$$\frac{1}{\lambda} = \frac{2.5\,\text{eV}}{1240\,\text{eV}\cdot\text{nm}} + \frac{1}{600\,\text{nm}} \approx 3.68 \times 10^{-3}\,\text{nm}^{-1} \tag{6}$$

$$\lambda \approx 272\,\text{nm} \tag{7}$$

2. Find the strength of the transverse magnetic field required to bend all the photoelectrons within a circle of 20 cm when light of wavelength 400 nm is incident on a barium emitter. The work function of barium is 2.5 eV.

Solution: The energies in the problem would imply electron velocities of the order $10^5 - 10^6$ m/s, so we may safely neglect relativistic effects. From classical electromagnetism, a particle of charge q and mass m traveling at velocity v at a right angle to a magnetic field follows circular motion of radius r:

$$r = \frac{mv}{qB} = \frac{p}{qB} \tag{8}$$

where p is the momentum of the particle. The photoelectric equation gives us the kinetic energy of the electrons, which we may also relate to the momentum:

$$K = e\Delta V = \frac{hc}{\lambda_i} - W = \frac{p^2}{2m} \tag{9}$$

where V is the stopping potential (not needed here), λ_i the incident photon wavelength, and W the work function. Solving the above for P, we find

$$p = \sqrt{2mK} = \sqrt{2m\left(\frac{hc}{\lambda_i} - W\right)} \tag{10}$$

Combining this with our expression for the radius,

$$r = \frac{\sqrt{2m\left(\frac{hc}{\lambda_i} - W\right)}}{qB} < r_o \tag{11}$$

where $r_o = 0.2$ m is the radius within we wish to confine the photoelectrons. Solving for B,

$$B > \frac{\sqrt{2m\left(\frac{hc}{\lambda_i} - W\right)}}{qr_o} \approx 13\,\mu\text{T} \tag{12}$$

3. Show that the relation between the directions of motion of the scattered photon and the recoiling electron in Compton scattering is

$$\frac{1}{\tan\left(\theta/2\right)} = \left(1 + \frac{hf_i}{m_e c^2}\right)\tan\varphi \tag{13}$$

Solution: Start with the momentum conservation equations for Compton scattering:

$$\alpha_i - \alpha_f \cos\theta = \left(\frac{p_e}{mc}\right)\cos\varphi \tag{14}$$

$$\alpha_f \sin\theta = \left(\frac{p_e}{mc}\right)\sin\varphi \tag{15}$$

where $\alpha_{i(f)} = hf_{i(f)}/mc^2$. Dividing them, we have

$$\tan\varphi = \frac{\alpha_f \sin\theta}{\alpha_i - \alpha_f \cos\theta} = \frac{\sin\theta}{\frac{\alpha_i}{\alpha_f} - \cos\theta} \tag{16}$$

We can use the Compton equation to substitute for α_i/α_f in terms of α_i alone:

$$\tan\varphi = \frac{\sin\theta}{\frac{\alpha_i}{\alpha_f} - \cos\theta} = \frac{\sin\theta}{1 + \alpha_i(1-\cos\theta) - \cos\theta} = \frac{\sin\theta}{(1+\alpha_i) - (1+\alpha_i)\cos\theta} \tag{17}$$

$$\tan\varphi = \frac{1}{1+\alpha_i}\frac{\sin\theta}{1-\cos\theta} \tag{18}$$

With the aid of a rather obscure trigonometric identity, we can simplify this further. Noting

$$\frac{1-\cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right) \tag{19}$$

we have

$$(1+\alpha_i)\tan\varphi = \frac{1}{\tan(\theta/2)} \qquad \text{or} \qquad \frac{1}{\tan(\theta/2)} = \left(1 + \frac{hf_i}{m_e c^2}\right)\tan\varphi \tag{20}$$

4. If the maximum energy imparted to an electron in Compton scattering is 45 keV, what is the wavelength of the incident photon?

Solution: The electron kinetic energy in Compton scattering (derived in the notes) is

$$E_e = mc^2\left(\frac{\alpha_i^2(1-\cos\theta)}{1+\alpha_i(1-\cos\theta)}\right) \tag{21}$$

which has a maximum at $\theta = 180°$, giving a maximum energy

$$E_{e,\mathrm{max}} = mc^2\left(\frac{2\alpha_i^2}{1+2\alpha_i}\right) = 45\,\mathrm{keV} \tag{22}$$

Solving this for α_i will yield the incident wavelength, since $\alpha_i = hf_i/mc^2 = hc/\lambda mc^2$. Let $\epsilon = E_{e,\text{max}}/mc^2$. Then

$$\epsilon = \left(\frac{2\alpha_i^2}{1+2\alpha_i}\right) \tag{23}$$

$$\epsilon\left(1+2\alpha_i\right) = \epsilon + 2\epsilon\alpha_i = 2\alpha_i^2 \tag{24}$$

$$0 = 2\alpha_i^2 - 2\epsilon\alpha_i - \epsilon \tag{25}$$

$$\alpha_i = \frac{2\epsilon \pm \sqrt{4\epsilon^2+8\epsilon}}{4} = \frac{1}{2}\epsilon \pm \frac{1}{2}\sqrt{\epsilon^2+2\epsilon} = \frac{1}{2}\epsilon\left(1+\sqrt{1+\frac{2}{\epsilon}}\right) = \frac{hc}{\lambda mc^2} \tag{26}$$

Noting $\epsilon = 45\,\text{keV}/511\text{ keV} \approx 0.088$ since $mc^2 \approx 511\,\text{keV}$ for an electron, and $hc \approx 1240\,\text{eV}\cdot\text{nm}$,

$$\frac{1}{\lambda_i} = \frac{mc^2}{2hc}\epsilon\left(1+\sqrt{1+\frac{2}{\epsilon}}\right) \approx 106.5\,\text{nm}^{-1} \tag{27}$$

$$\lambda \approx 9.4\times10^{-3}\,\text{nm} = 9.4\times10^{-12}\,\text{m} \tag{28}$$

5. Show that a free electron at rest cannot absorb a photon, and hence Compton scattering *must* occur with free electrons. *Hint:* try to conserve energy and momentum.

Solution: All we really need to do is conserve energy and momentum for photon absorption by a stationary, free electron and show that something impossible is implied. Before the collision, we have a photon of energy hf and momentum h/λ and an electron with rest energy mc^2. Afterward, we have an electron of energy $(\gamma-1)+mc^2 = \sqrt{p^2c^2+m^2c^4}$ (i.e., the afterward the electron has acquired kinetic energy, but retains its rest energy) and momentum $p_e = \gamma mv$. Momentum conservation dictates that the absorbed photon's entire momentum be transferred to the electron, which means it must continue along the same line that the incident photon traveled. This makes the problem one dimensional, which is nice.

Enforcing conservation of energy and momentum, we have:

$$(\text{initial}) = (\text{final}) \tag{29}$$

$$hf + mc^2 = \sqrt{p^2c^2+m^2c^4} \qquad \text{energy conservation variant 1} \tag{30}$$

$$hf + mc^2 = (\gamma-1)\,mc^2 \qquad \text{energy conservation variant 2} \tag{31}$$

$$\frac{h}{\lambda} = p_e = \gamma mv \qquad \text{momentum conservation} \tag{32}$$

From this point on, we can approach the problem in two ways, using either expression for the electron's energy. We'll do both, just to give you the idea. First, we use conservation of momentum to put the electron momentum in terms of the photon frequency:

$$\frac{h}{\lambda} = p_e \qquad \Longrightarrow \qquad \frac{hc}{\lambda} = hf = p_e c \tag{33}$$

Now substitute that in the first energy conservation equation to eliminate p_e, square both sides, and collect terms:

$$\left(hf + mc^2\right)^2 = \left(\sqrt{p^2c^2 + m^2c^4}\right)^2 = \left(\sqrt{h^2f^2 + m^2c^4}\right)^2 \tag{34}$$

$$h^2f^2 + 2hfmc^2 + m^2c^4 = h^2f^2 + m^2c^4 \tag{35}$$

$$2hfmc^2 = 0 \qquad \Longrightarrow \qquad f = 0 \qquad \Longrightarrow \qquad p_e = v = 0 \tag{36}$$

Thus, we conclude that the only way a photon can be absorbed by the stationary electron is if its frequency is zero, *i.e., if there is no photon to begin with!* Clearly, this is silly.

We can also use the second variant of the conservation of energy equation along with momentum conservation to come to an equally ridiculous conclusion:

$$hf = \frac{hc}{\lambda} = (\gamma - 1)\,mc^2 \qquad \text{energy conservation variant 2} \tag{37}$$

$$\frac{h}{\lambda} = \gamma m v \qquad \text{or} \qquad \frac{hc}{\lambda} = \gamma m v c \qquad \text{momentum conservation} \tag{38}$$

$$\Longrightarrow \qquad \gamma m v c = (\gamma - 1)\,mc^2 \tag{39}$$

$$(\gamma - 1)\,c = \gamma v \tag{40}$$

$$\frac{\gamma - 1}{\gamma} = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \text{(definition of } \gamma\text{)} \tag{41}$$

$$\left(\frac{\gamma - 1}{\gamma}\right)^2 = 1 - \frac{1}{\gamma^2} \tag{42}$$

$$\gamma^2 - 2\gamma + 1 = \gamma^2 - 1 \tag{43}$$

$$\gamma = 1 \qquad \Longrightarrow \qquad v = 0 \tag{44}$$

Again, we find an electron recoil velocity of zero, implying zero incident photon frequency, which means there is no photon in the first place! Conclusion: stationary electrons *cannot* absorb photons, but they can Compton scatter them.

6. Determine the maximum scattering angle in a Compton experiment for which the scattered photon can produce a positron-electron pair. *Hint:* twice the electron's rest energy is required of the incident photon, see `http://en.wikipedia.org/wiki/Pair_production`.

Solution: All this means is that the exiting (scattered) photon must have an energy of at least

$2mc^2$. In terms of the dimensionless photon energies $\alpha_i = hf_i/mc^2$, $\alpha_f = hf_f/mc^2$, the Compton equation reads

$$\frac{1}{\alpha_i} = \frac{1}{\alpha_f} - (1 - \cos\theta) \tag{45}$$

If the exiting photon energy is $hf_f = 2mc^2$, this means $\alpha_f = 2$. Solving the Compton equation for α_i,

$$\alpha_i = \frac{1}{\frac{1}{\alpha_i} - (1 - \cos\theta)} \tag{46}$$

Physically, α_i is an energy and it must be positive – that is the most basic requirement we can make. In the equation above, the numerator is clearly always positive, so the only condition we can enforce is that the denominator remain positive. This requires

$$\frac{1}{\alpha_i} > (1 - \cos\theta) \tag{47}$$

If the denominator tends toward zero, α_i tends toward infinity, so this is equivalent to requiring that the incident photon have finite energy – also very sensible! Solving for θ,

$$\cos\theta > 1 - \frac{1}{\alpha_i} \tag{48}$$

$$\theta < \cos^{-1}\left(1 - \frac{1}{\alpha_i}\right) \tag{49}$$

In the last line, we reverse the inequality because $\cos\theta$ is a decreasing function of θ as θ increases from 0. This amounts to

$$\theta < \cos^{-1}\left(1 - \frac{1}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60° \tag{50}$$

7. In Compton scattering what is the kinetic energy of the electron scattered at an angle φ with respect to the incident photon?

Solution: One way is simply to use the electron's energy derived in the notes and the result of problem 3. In principle, that is it: one has the energy in terms of θ, and a way to get θ from φ, so the energy can be determined from a knowledge of α_i and φ alone. This is acceptable, but inelegant. Finding a direct relationship between energy, α_i, and φ would be much nicer.

Start with the electron energy derived in the notes, with $\epsilon = E_e/mc^2$:

$$\epsilon = \frac{\alpha_i^2 \left(1 - \cos\theta\right)}{1 + \alpha_i \left(1 - \cos\theta\right)} \tag{51}$$

First may use the trigonometric identity $1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right)$:

$$\epsilon = \frac{\alpha_i^2 \left(2\sin^2\left(\frac{\theta}{2}\right)\right)}{1 + \alpha_i \left(2\sin^2\left(\frac{\theta}{2}\right)\right)} = \frac{2\alpha_i^2 \sin^2\left(\frac{\theta}{2}\right)}{1 + 2\alpha_i \sin^2\left(\frac{\theta}{2}\right)} \tag{52}$$

With one more identity, we can put this in terms of $\tan\left(\frac{\theta}{2}\right)$, at which point we can use the result of problem 3. The next identity is:

$$\sin^2\theta = \frac{\tan^2\theta}{1 + \tan^2\theta} \tag{53}$$

This is indeed a strange identity, and to derive it we should recall $\sec^2\theta = 1 + \tan^2\theta$. Working it backwards:

$$\frac{\tan\theta}{\sqrt{1 + \tan^2\theta}} = \frac{\sin\theta}{\cos\theta}\frac{1}{\sec\theta} = \frac{\sin\theta}{\cos\theta}\cos\theta = \sin\theta \tag{54}$$

which yields

$$\epsilon = \frac{2\alpha_i^2 \left(\dfrac{\tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}\right)}{1 + 2\alpha_i \left(\dfrac{\tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right)}\right)} = \frac{2\alpha_i^2 \tan^2\left(\frac{\theta}{2}\right)}{1 + \tan^2\left(\frac{\theta}{2}\right) + 2\alpha_i \tan^2\left(\frac{\theta}{2}\right)} = \frac{2\alpha_i^2}{\dfrac{1}{\tan^2\left(\frac{\theta}{2}\right)} + 1 + 2\alpha_i} \tag{55}$$

Problem 3 gives us

$$\frac{1}{\tan\left(\theta/2\right)} = \left(1 + \alpha_i\right)\tan\varphi \tag{56}$$

Using this identity, we have the electron energy in terms of φ and α_i alone:

$$\epsilon = \frac{2\alpha_i^2}{1 + 2\alpha_i + \left(1 + \alpha_i\right)^2 \tan^2\varphi} \tag{57}$$

or

$$E_e = mc^2 \left(\frac{2\alpha_i^2}{1 + 2\alpha_i + (1+\alpha_i)^2 \tan^2 \varphi} \right) \tag{58}$$

8. A radio station broadcasts at a frequency of 1 MHz with a total radiated power of 5 kW. **(a)** What is the wavelength of this radiation? **(b)** What is the energy (in electron volts) of the individual quanta that compose the radiation? How many photons are emitted per second? Per cycle of oscillation? **(c)** A certain radio receiver must have 2 μW of radiation power incident on its antenna in order to provide an intelligible reception. How many 1 MHz photons does this require per second? Per cycle of oscillation? **(d)** Do your answers for parts (b) and (c) indicate that the granularity of electromagnetic radiation can be neglected in these circumstances?

Solution: **(a)** Radio waves are just light, so knowledge of the frequency gives us the wavelength:

$$\lambda = \frac{c}{f} = 300\,\mathrm{m} \tag{59}$$

(b) The energy of an individual photon is just $hf = 4.1 \times 10^{-9}\,\mathrm{eV} = 6.63 \times 10^{-28}\,\mathrm{J}$. The station's power ($P$) is the energy ($\Delta E$) per unit time ($\Delta t$) emitted, and must just be the energy per photon times the number of photons per unit time. If we call the number of photons per unit time $\Delta N/\Delta t$,

$$P = \frac{\Delta E}{\Delta t} = hf\frac{\Delta N}{\Delta t} \qquad \Longrightarrow \qquad \frac{\Delta N}{\Delta t} = \frac{P}{hf} \approx 7.5 \times 10^{30}\,\mathrm{photons/s} \tag{60}$$

There are 10^6 periods of oscillation per second, so that means that there are approximately 7.5×10^{24} photons/period being emitted.

(c) This is precisely the same as the previous question, except the relevant power is 2 μW instead of 5000 W.

$$\frac{\Delta N}{\Delta t} = \frac{P}{hf} \approx 3.0 \times 10^{21}\,\mathrm{photons/s} \tag{61}$$

Again, there are 10^6 periods of oscillation per second, so there are approximately 3.0×10^{15} photons/period being emitted.

This is certainly enough photons that the granularity of electromagnetic radiation is utterly negligible for everyday power levels such as these.

What *would* the power level have to be for 1 MHz photons to have a noticeable granularity? Roughly speaking, the sampling theorem says that if a function $x(t)$ contains no frequencies higher than B, it is completely determined by sampling at a rate of $1/2B$.[i]. We could say then that the granularity in a signal would be noticeable in this case if the photons were coming at less than 2 per cycle of oscillation. That means

$$\frac{\Delta N}{\Delta t} = \frac{P}{hf} \approx 2\,\text{photons/period} = 2 \times 10^6\,\text{photons/sec} \tag{62}$$

With the given photon frequency of 1 MHz, we find $P \sim 10^{-21}$ W, a negligible amount of power. For photons of visible light, in the 10^{15} Hz range, the power is $\sim 10^{-12}$ W, which is close to the limit of human vision. With dark-adapted scotopic vision, we detect about $8 \times 10^{-11}\,\text{W/m}^2$ of green light (550 nm), which means down to around $\sim 10^2 - 10^3$ photons/s for an average-sized eye. Just about enough to notice the granularity, but not quite.[ii]

9. *Time delay in the photoelectric effect.* A beam of ultraviolet light of intensity 1.6×10^{-12} W is suddenly turned on and falls on a metal surface, ejecting electrons through the photoelectric effect. The beam has a cross-sectional area of $1\,\text{cm}^2$, and the wavelength corresponds to a photon energy of 10 eV. The work function of the metal is 5 eV. How soon might one expect photoelectric emission to occur? *Note:* $1\,\text{eV} = 1.6 \times 10^{-19}$ J.

(a) One classical model suggests an estimate based on the time needed for the work function energy (5 eV) to be accumulated over the area of one atom (radius ~0.1 nm). Calculate how long this would be, assuming the energy of the light beam to be uniformly distributed over its cross section.

(b) Actually, as Lord Rayleigh showed in 1916, the estimate from (a) is too pessimistic. An atom can present an effective area of about λ^2 to light of wavelength λ corresponding to its resonance frequency. Calculate a time delay on this basis.

(c) On the quantum picture of the process, it is possible for photoelectron emission to begin immediately – as soon as the first photon strikes the emitting surface. But to obtain a time that may be compared to the classical estimates, calculate the *average* time interval between arrival of successive 10 eV photons. This would also be the average time delay between switching on the source and getting the first photoelectron. *Hint:* think of the power as photons per unit time.

Solution: The power absorbed by the atom is the fraction of the beam's total area that it intercepts times the total power in the beam. If the beam has power P_b and area A_b, and a circular atom of

[i] `http://en.wikipedia.org/wiki/Nyquist-Shannon_sampling_theorem`

[ii] Actually, it is more complicated than this. The sensors in the eye are capable of detecting single photons, but our neural hardware filters the incoming signals to smooth out this granularity. If it didn't, we would be too distracted by the granularity in low light. See `http://math.ucr.edu/home/baez/physics/Quantum/see_a_photon.html` for a nice discussion.

radius r has an area πr^2, the power absorbed by the atom P_a is

$$P_a = P_b \frac{\pi r^2}{A_b} \tag{63}$$

If the beam power is constant, then so is the power absorbed by the atom. Constant power means constant energy per unit time, so the amount of energy ΔE absorbed in a time Δt by the atom is $\Delta E = P_a \Delta t$, or

$$\Delta t = \frac{\Delta E}{P_a} = \frac{\Delta E A_b}{P_b \pi r^2} \tag{64}$$

The atom needs to absorb an energy of $\Delta E = 5\,\text{eV}$, which will require $\Delta t \approx 1.6 \times 10^9\,\text{s} \sim 50\,\text{yr}$ using the information given. I have it on good authority that this experiment is easily completed in the PH255 laboratory in a few minutes, so something has gone horribly wrong.

Lord Rayleigh used a more accurate cross-section (recall our discussion of cross sections when we analyzed radiation) of λ^2, which in terms of the light energy E is

$$\lambda^2 = \left(\frac{hc}{E}\right)^2 = A_a \tag{65}$$

This leads to

$$\Delta t = \frac{\Delta E}{P_a} = \frac{\Delta E A_b}{P_b A_a} = \frac{\Delta E A_b E^2}{P_b h^2 c^2} \approx 3200\,\text{s} \sim 1\,\text{hr} \tag{66}$$

Better, but still very much wrong.

In the quantum model, the power in the beam is just the number of photons per second times the energy per photon. If we call the number of photons per unit time $\Delta N/\Delta t$, and the energy per photon E

$$P = \frac{\Delta E}{\Delta t} = E \frac{\Delta N}{\Delta t} = 1.6 \times 10^{-12}\,\text{W} \tag{67}$$

This implies $\frac{\Delta N}{\Delta t} \approx 10^6$ photons/sec, or that *on average*, 10^{-6} seconds passes between photons to account for 10^6 arriving over the course of one second.