University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair Fall 2010

Problem Set 4: Uncertainty, Wave Mechanics

Instructions:

1. Answer all questions below. Show your work for full credit.
2. All problems are due Fri 17 September 2010 by the end of the day.
3. You may collaborate, but everyone must turn in their own work.

1. **(a)** Determine the accelerating potential necessary to give an electron a de Broglie wavelength of 0.1 nm, which is the size of the interatomic spacing of atoms in a crystal. **(b)** If we wish to observe an object which is 0.25 nm in size, what is the minimum-energy photon which can be used?

2. From the relationship $\Delta p \Delta x \geqslant h/4\pi$, show that for a particle moving in a circle $\Delta L\, \Delta\theta \geqslant h/4\pi$. The quantity ΔL is the uncertainty in angular momentum and $\Delta\theta$ is the uncertainty in the angle.

3. The position of a particle is measured by passing it through a slit of width d. Find the corresponding uncertainty induced in the particle's momentum.

4. *Upper limit on the rest mass of the photon.* de Broglie placed an upper limit of 10^{-47} kg on the rest mass of a photon by assuming that radio waves of wavelength 30 m travel with a speed of at least 99% the speed of visible light ($\lambda = 500\,\text{nm}$). Beginning with the equation $E = hf = \gamma mc^2$ for a photon of rest mass m, obtain an exact expression for v/c in terms of mc^2 and hf. Use this to find an approximate expression for $(c-v)/c$ in the case $mc^2 \ll hf$. Check de Broglie's calculation of the 10^{-47} kg limit.

5. Consider the classical motion of a pendulum bob which, for small amplitudes of oscillation, moves effectively as a harmonic oscillator along a horizontal axis according to the equation $x(t) = A \sin \omega t$. The probability that the bob will be found within a small distance Δx at x in random observations is proportional to the time it spends in this region during each swing. Obtain a mathematical expression for this probability as a function of x ($P(x)$), assuming $\Delta x \ll A$. (*Note: the probability of the bob being* somewhere *must be* 1, *so* $\int_{-A}^{A} P(x)\, dx = 1$. This is a good double check.)

6. *Zero point energy of a harmonic oscillator.* The frequency f of a harmonic oscillator of mass m and elasticity constant k is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{1}$$

The energy of the oscillator is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{2}$$

where p is the system's linear momentum and x is the displacement from its equilibrium position. Use the uncertainty principle, $\Delta x \Delta p \approx \hbar/2$, to express the oscillator's energy E in terms of x and show, by taking the derivative of this function and setting $dE/dx = 0$, that the minimum energy of the oscillator (its ground state energy) is $E_{\min} = hf/2$.

7. Consider the experimental setup sketched below, whose purpose is to measure the position of an electron. Electrons are in a beam having well-defined momentum p_x along the x axis. The microscope (lens + screen) is to be used to see where the electron is located by viewing the light scattered off of the electron. We shine a light (wavelength λ) along the x axis, a photon will scatter off of an electron, and the photon will recoil through the microscope. The resolution of this microscope gives the precision to which the electron's position can be determined, and is known from optics:

$$\Delta x \sim \frac{\lambda}{\sin\varphi} \tag{3}$$

It seems that if we make λ small enough, and $\sin\varphi$ large enough, Δx can be made as small as desired. However, we will have sacrificed knowledge of the electron's recoil momentum, since we can only determine the (equal and opposite) photon recoil momentum to within the angle subtended by the aperture φ.

Estimate the uncertainty in the x component of the recoil momentum of the electron Δp_x, and show that the uncertainty principle is obeyed in this microscope.

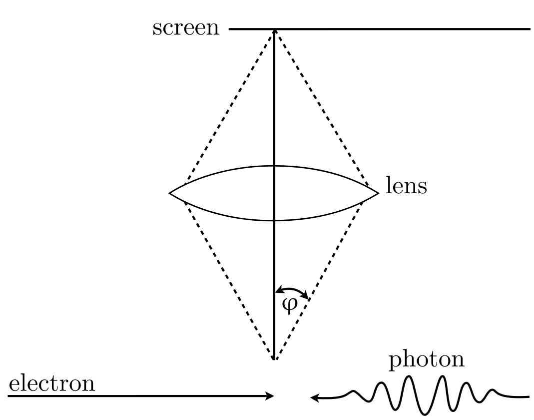

Figure 1: *Schematic drawing of the Heisenberg microscope for the measurement of electron position.*

8. In quantum physics, both the position and momentum can be separately described by their own wave functions which are related by a Fourier transformation. If the position wave function is $\psi(x)$, and the momentum wave function $\varphi(k)$, where $p = \hbar k$,

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_k \varphi(k) e^{ikx}\, dk \tag{4}$$

where the integral is over all $k \in [-\infty, \infty]$. Consider a rectangular pulse, given by

$$\varphi(k) = \begin{cases} 0 & k < -K \\ N & -K < k < K \\ 0 & K < k \end{cases} \tag{5}$$

(a) Find the position wave function $\psi(x)$.[i]
(b) Find the value of N for which[ii]

$$\int_{-\infty}^{\infty} |\psi(x)|^2\, dx = 1 \tag{6}$$

(c) How is this related to the choice of N for which

$$\int_{-\infty}^{\infty} |\varphi(x)|^2\, dk = \frac{1}{2\pi} \tag{7}$$

(d) Show that a reasonable definition of Δx for your answer to (a) yields

$$\Delta k \Delta x > 1 \qquad \text{or} \qquad \Delta p \Delta x > \hbar \tag{8}$$

independent of the value of K.

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[i]Recall the Euler identity $e^{ikx} = \cos kx + i \sin kx$.
[ii]Note $\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$ and $\int_0^\infty \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}$.