University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair Fall 2010

Problem Set 4: Solutions

1. (a) Determine the accelerating potential necessary to give an electron a de Broglie wavelength of 0.1 nm, which is the size of the interatomic spacing of atoms in a crystal. **(b)** If we wish to observe an object which is 0.25 nm in size, what is the minimum-energy photon which can be used?

Solution: **(a)** The de Broglie wavelength of the electron is $\lambda = h/p$. If an electron is accelerated through a potential of ΔV, it loses a potential energy $e\Delta V$ and acquires a kinetic energy $K = p^2/2m = e\Delta V$. Thus, the electron's momentum is

$$p = \sqrt{2me\Delta V} = \frac{h}{\lambda} \tag{1}$$

Solving for the potential difference,

$$\Delta V = \frac{h^2}{2me\lambda^2} \approx 151\,\text{V} \tag{2}$$

(b) In order to observe an object of 0.25 nm size, we need a photon of at least that wavelength (or smaller): $\lambda \leqslant 0.25\,\text{nm}$. The photon energy is $E = hf = hc/\lambda$, so

$$E \geqslant \frac{hc}{\lambda} \approx 4.96\,\text{keV} \tag{3}$$

2. From the relationship $\Delta p \Delta x \geqslant h/4\pi$, show that for a particle moving in a circle $\Delta L\,\Delta\theta \geqslant h/4\pi$. The quantity ΔL is the uncertainty in angular momentum and $\Delta\theta$ is the uncertainty in the angle.

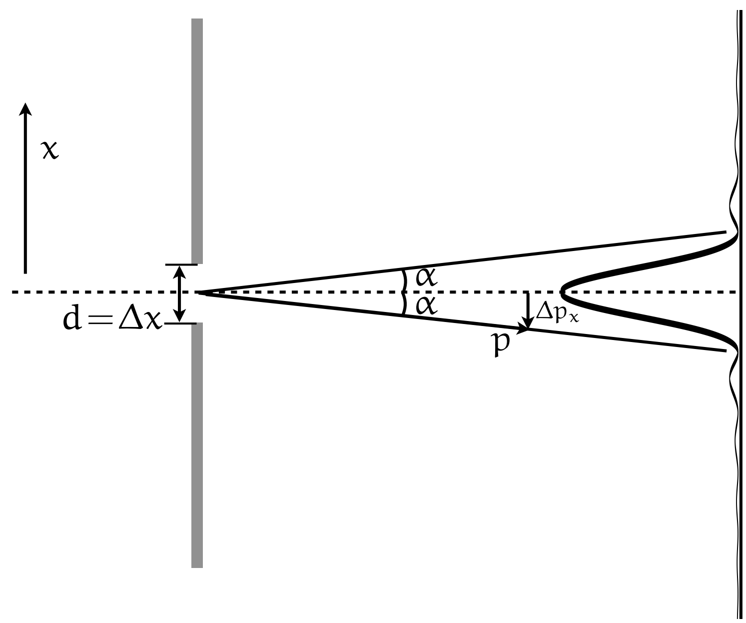
Solution: An electron traveling in a circular path of radius r covering a distance Δx along the circle's perimeter moves through an angle $\Delta\theta$ according to the arclength formula $\Delta x = \Delta\theta r$. Since r is fixed, this formula holds equally well for an uncertainty in position. The momentum of a the particle is $p = mv = mr\omega$, and given that m and r are constants, any uncertainty Δp in the particle's momentum must come from an uncertainty in ω, $\Delta p = mr\Delta\omega$.

On the other hand, the particle's angular momentum is $L = mvr = mr^2\omega$, and its uncertainty can only come from an uncertainty in ω, $\Delta L = mr^2\Delta\omega$. Putting our relationships together,

$$\Delta x\,\Delta p = (r\Delta\theta)\,(mr\Delta\omega) = \Delta\theta\left(mr^2\Delta\omega\right) = \Delta\theta\,\Delta L \geqslant \frac{h}{4\pi} \tag{4}$$

3. The position of a particle is measured by passing it through a slit of width d. Find the corresponding uncertainty induced in the particle's momentum.

Solution: A sketch may help:



When the de Broglie wave corresponding to the particle passes through the slit, it will be diffracted showing an intensity pattern like that shown above. Most of the diffracted intensity on a vertical screen will be between the first points of zero intensity at angles α and $-\alpha$, the central region corresponding to the "direct beam." These angles related to the slit width d and de Broglie wavelength according to the relationship given from diffraction (wave optics) theory:

$$\sin\alpha = \frac{\lambda}{d} \tag{5}$$

Once the particle is diffracted through the slit, it will acquire some unknown momentum in the vertical (x) direction. Although we don't know exactly where the particle will hit the screen, it is by far most probable that it lies in the central region, with its equivalent straight-line path making at most an angle α with respect to the horizontal. Since the particle's initial momentum is $p = h/\lambda$, this means that the maximum vertical component of its momentum is

$$\Delta p_x = p \sin\alpha = \frac{h}{\lambda}\frac{\lambda}{d} = \frac{h}{d} \tag{6}$$

The uncertainty in the particle's vertical position arises because we do not know exactly where the particle went through the slit, just that it made it through the slit somewhere over its width d. Thus, $\Delta x = d$, so the uncertainty in position can be made as small as we like by making d smaller. However, we see above that this would make the uncertainty in momentum larger by the same amount, and

$$\Delta x\,\Delta p = h \tag{7}$$

The uncertainty relationship is obeyed, *independent* of the slit's width or the particle's wavelength.

4. *Upper limit on the rest mass of the photon.* de Broglie placed an upper limit of 10^{-47} kg on the rest mass of a photon by assuming that radio waves of wavelength 30 m travel with a speed of at least 99% the speed of visible light ($\lambda = 500$ nm). Beginning with the equation $E = hf = \gamma mc^2$ for a photon of rest mass m, obtain an exact expression for v/c in terms of mc^2 and hf. Use this to find an approximate expression for $(c-v)/c$ in the case $mc^2 \ll hf$. Check de Broglie's calculation of the 10^{-47} kg limit.

Solution: We presume that the photon energy is due to some rest mass-energy plus kinetic energy, so we can write its total energy as we would any other particle, $E = \gamma mc^2$. Since the photon energy is also hf, we have

$$hf = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{8}$$

$$\frac{v}{c} = \sqrt{1 - \left(\frac{mc^2}{hf}\right)^2} \tag{9}$$

If the rest mass-energy is small, such that $mc^2 \ll hf$, then we may use the approximation $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ when $x \ll 1$:

$$\frac{v}{c} \approx 1 - \frac{1}{2}\left(\frac{mc^2}{hf}\right)^2 \tag{10}$$

$$\frac{c-v}{c} = 1 - \frac{v}{c} \approx \frac{1}{2}\left(\frac{mc^2}{hf}\right)^2 \tag{11}$$

Solving this for m, we have

$$m = \frac{hf}{c^2}\sqrt{2\left(\frac{c-v}{c}\right)} \tag{12}$$

If radio waves with $\lambda = 30\,\text{m}$ travel at $0.99c$, then $(c-v)/c = 0.01$. Using $\lambda f = v = 0.99c$,

$$m = \frac{hv}{\lambda c^2}\sqrt{0.02} = \frac{0.99h}{\lambda c}\sqrt{0.02} \approx 1\times 10^{-44}\,\text{kg} \tag{13}$$

With this data, de Broglie was being a bit overzealous, but not by much ...

5. Consider the classical motion of a pendulum bob which, for small amplitudes of oscillation, moves effectively as a harmonic oscillator along a horizontal axis according to the equation $x(t) = A\sin\omega t$. The probability that the bob will be found within a small distance Δx at x in random observations is proportional to the time it spends in this region during each swing. Obtain a mathematical expression for this probability as a function of x ($P(x)$), assuming $\Delta x \ll A$. (*Note: the probability of the bob being* somewhere *must be* 1*, so* $\int_{-A}^{A} P(x)\,dx = 1$*.* This is a good double check.)

Solution: The probability that the bob is in a tiny region dx in the interval $[x, x+dx]$ is

$$P(\text{in }[x, x+dx])\,dx = \frac{\text{time to move }dx}{\text{half the period}} \tag{14}$$

since it takes half the period of motion to go from one extremal point to the other, or from $x = -A$ to $x = A$. Here the speed is a function of position, $v(x)$, so to travel a distance dx requires a time given by

$$dx = v(x)\,dt \qquad \text{or} \qquad dt = \frac{dx}{v(x)} \tag{15}$$

Thus, the probability is

$$P(x)\,dx = \frac{dt}{\frac{1}{2}T} = \frac{2}{T}\frac{dx}{v(x)} \qquad \text{or} \qquad P(x) = \frac{2}{T}\frac{1}{v(x)} \tag{16}$$

The velocity can be found from the kinetic energy: since $K = \frac{1}{2}mv^2$, then $v = \sqrt{2K/m}$. Our pendulum follows simple harmonic motion, and as such, it really wouldn't matter if it were an ideal pendulum, a a mass connected to a spring, or any other kind of simple harmonic oscillator. For any simple harmonic oscillator, the total energy is always $E_{tot} = \frac{1}{2}kA^2$, where k is a force constant and A the amplitude of motion. For a pendulum, we make the identification that

$$\frac{k}{m} = \frac{g}{L} \tag{17}$$

where g is the gravitational acceleration and L the length of the pendulum. Kinetic energy is total energy minus potential, so conservation of energy gives

$$K = E_{tot} - U \tag{18}$$

$$v = \sqrt{\frac{2\left(E_{tot} - U\right)}{m}} \tag{19}$$

Noting that the period of a simple harmonic oscillator is $T = 2\pi\sqrt{k/m}$, the probability is then

$$P(x) = \frac{2}{T}\sqrt{\frac{m}{2\left(E_{tot} - U\right)}} \tag{20}$$

For a simple harmonic oscillator, $T = 2\pi\sqrt{k/m}$, $E_{tot} = \frac{1}{2}kA^2$, and the potential energy of the system at any position x is $U = \frac{1}{2}kx^2$, which gives

$$P(x) = \frac{1}{\pi}\sqrt{\frac{m}{k}}\sqrt{\frac{m}{kA^2 - kx^2}} = \frac{m}{\pi k}\frac{1}{\sqrt{A^2 - x^2}} \tag{21}$$

This does not necessarily satisfy the condition that the probability of the pendulum being *somewhere* is unity, we must enforce that

$$1 = \int\limits_{-\infty}^{\infty} P(x)\,dx \tag{22}$$

This procedure is called *normalization*, and all it does is ensure that our probability distribution is logically consistent – this is a procedure we will go over in more depth in future lectures. If you didn't do this, that's fine for now, and no points will be taken off – it is not something we discussed much yet, and your answer is proportionally right without normalization. If we give our probability distribution an overall constant multiplying factor C, we can use the condition above to enforce unit probability integrated over all possible x. Noting that our pendulum can only take values of $x \in [-A, A]$

$$1 = \int_{-\infty}^{\infty} P(x)\,dx = \int_{-A}^{A} \frac{Cm}{\pi k}\frac{1}{\sqrt{A^2 - x^2}}\,dx = \left[\frac{Cm}{\pi k}\tan^{-1}\left(\frac{x\sqrt{A^2 - x^2}}{A^2 - x^2}\right)\right]_{-A}^{A} = Cmk \tag{23}$$

This gives $C = k/m$, so our properly normalized probability distribution is

$$P(x) = \frac{1}{\pi}\frac{1}{\sqrt{A^2 - x^2}} \tag{24}$$

There is an interesting article on this problem in the American Journal of Physics (vol. 63, page 823, 1995), a physics education journal. It is available online from campus (subscription screened by IP address).

6. *Zero point energy of a harmonic oscillator.* The frequency f of a harmonic oscillator of mass m and elasticity constant k is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{25}$$

The energy of the oscillator is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{26}$$

where p is the system's linear momentum and x is the displacement from its equilibrium position. Use the uncertainty principle, $\Delta x \Delta p \approx \hbar/2$, to express the oscillator's energy E in terms of x and show, by taking the derivative of this function and setting $dE/dx = 0$, that the minimum energy of the oscillator (its ground state energy) is $E_{\text{min}} = hf/2$.

Solution: The minimum uncertainty in momentum Δp, given an uncertainty Δx in position is given by the uncertainty principle:

$$\Delta x \Delta p = \frac{\hbar}{2} \qquad \Longrightarrow \qquad \Delta p = \frac{\hbar}{2\Delta x} \tag{27}$$

The minimum uncertainty is also then the minimum average value we can expect either variable to take on: $p_{\text{min}} = \Delta p \equiv p$, $x_{\text{min}} = \Delta x \equiv x$. The energy equation may the be rewritten in terms of the minimal x and p:

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}kx^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}m\omega^2x^2 \tag{28}$$

In the last line, we used $\omega = \sqrt{k/m}$, so $k = m\omega^2$. Minimizing the energy with respect to x,

$$\frac{dE}{dx} = \frac{-2\hbar^2}{4mx^3} + m\omega^2 x = 0 \qquad \Longrightarrow \qquad x^2 = \frac{\hbar}{2m\omega} \tag{29}$$

Plugging this back in to the energy equation, we have the minimum energy:

$$E_{\text{min}} = \frac{\hbar^2}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega} = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega \tag{30}$$

7. Consider the experimental setup sketched below, whose purpose is to measure the position of an electron. Electrons are in a beam having well-defined momentum p_x along the x axis. The microscope (lens + screen) is to be used to see where the electron is located by viewing the light scattered off of the electron. We shine a light (wavelength λ) along the x axis, a photon will scatter off of an electron, and the photon will recoil through the microscope. The resolution of this microscope gives the precision to which the electron's position can be determined, and is known from optics:

$$\Delta x \sim \frac{\lambda}{\sin\varphi} \tag{31}$$

It seems that if we make λ small enough, and $\sin\varphi$ large enough, Δx can be made as small as desired. However, we will have sacrificed knowledge of the electron's recoil momentum, since we can only determine the (equal and opposite) photon recoil momentum to within the angle subtended by the aperture φ.

Estimate the uncertainty in the x component of the recoil momentum of the electron Δp_x, and show that the uncertainty principle is obeyed in this microscope.

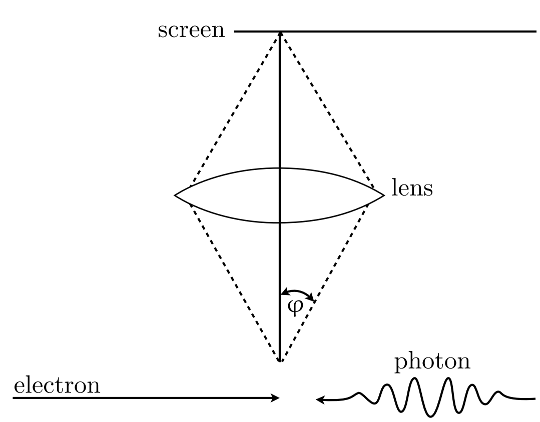


Figure 1: *Schematic drawing of the Heisenberg microscope for the measurement of electron position.*

Solution: This is basically the same problem as number 3, made slightly more realistic by the addition of a lens.

The electron will recoil after colliding with the photon, and we only know that the collision was such that the photon came off at an angle somewhere between φ and $-\varphi$, but we can't say any more than that. The horizontal component of the electron's recoil momentum can be anywhere from $p \sin \varphi$ to $-p \sin \varphi$, where p is the photon momentum. This gives an uncertainty in the electron's x momentum, and the microscope lens has its own uncertainty (resolution) given above.

The electron's recoil momentum will be equal and opposite to the photon's initial momentum h/λ, and it can have any angle between φ and $-\varphi$, so its x component can be anywhere in the range

$$\Delta p_x = 2p \sin \varphi = \frac{2h}{\lambda} \sin \varphi \tag{32}$$

Using the given uncertainty in position, a result from wave optics,

$$\Delta p_x \, \Delta x = \left(\frac{2h}{\lambda} \sin \varphi \right) \left(\frac{\lambda}{\sin \varphi} \right) = 2h \tag{33}$$

Since this is greater than the minimum uncertainty $h/4\pi$, the uncertainty principle is obeyed in this situation.

8. In quantum physics, both the position and momentum can be separately described by their own wave functions which are related by a Fourier transformation. If the position wave function is $\psi(x)$, and the momentum wave function $\varphi(k)$, where $p = \hbar k$,

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_k \varphi(k) e^{ikx} \, dk \tag{34}$$

where the integral is over all $k \in [-\infty, \infty]$. Consider a rectangular pulse, given by

$$\varphi(k) = \begin{cases} 0 & k < -K \\ N & -K < k < K \\ 0 & K < k \end{cases} \tag{35}$$

(a) Find the position wave function $\psi(x)$.[i]

[i] Recall the Euler identity $e^{ikx} = \cos kx + i \sin kx$.

(b) Find the value of N for which[ii]

$$\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1 \tag{36}$$

(c) How is this related to the choice of N for which

$$\int_{-\infty}^{\infty} |\varphi(x)|^2 \, dk = \frac{1}{2\pi} \tag{37}$$

(d) Show that a reasonable definition of Δx for your answer to (a) yields

$$\Delta k \Delta x > 1 \qquad \text{or} \qquad \Delta p \Delta x > \hbar \tag{38}$$

independent of the value of K.

Solution: The position wave function is:

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi(k) e^{ikx} \, dk = \frac{1}{\sqrt{2\pi}} \int_{-K}^{K} N e^{ikx} \, dk = \frac{N}{ix\sqrt{2\pi}} e^{ikx} \Big|_{-K}^{K} = \frac{N}{ix\sqrt{2\pi}} \left(e^{iKx} - e^{-iKx} \right) \\ &= \frac{N}{ix\sqrt{2\pi}} \left(\cos Kx + i \sin Kx - \cos -Kx - i \sin -Kx \right)\end{aligned} \tag{39}$$

For the last line, we used the Euler identity $e^{\theta} = \cos\theta + i\sin\theta$. Now note that $\cos x = \cos(-x)$ and $\sin x = -\sin(-x)$,

$$\psi(x) = \frac{N}{ix\sqrt{2\pi}} 2i \sin Kx = N\sqrt{\frac{2}{\pi}} \frac{\sin Kx}{x} \tag{40}$$

This is again a normalization procedure, like in problem 5. The integral can be identified with that given in the tiny footnote below by using the substitution $u = Kx$, $du = K\,dx$:

$$1 = \int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = \int_{-\infty}^{\infty} \frac{2N^2}{\pi} \frac{\sin^2 Kx}{x^2} \, dx = \frac{2N^2 K}{\pi} \int_{-\infty}^{\infty} \frac{\sin^2 Kx}{x^2} \, dx \tag{41}$$

[ii]Note $\int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2}$ and $\int_0^{\infty} \frac{\sin^2 x}{x^2} \, dx = \frac{\pi}{2}$.

Noting that the function $\sin^2 x/x$ is symmetric about $x=0$,

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = 2\int_{0}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi \tag{42}$$

and we have

$$1 = 2N^2 K \qquad \Longrightarrow \qquad N = \frac{1}{\sqrt{2K}} \tag{43}$$

Integrating $\varphi(k)$ over all k is trivial:

$$\frac{1}{2\pi} = \int\limits_{-\infty}^{\infty} |\varphi(x)|^2\, dk = \int\limits_{-K}^{K} N^2\, dk = 2N^2K \qquad \Longrightarrow \qquad N = \frac{1}{2\sqrt{\pi K}} \tag{44}$$

The two normalization factors for ψ and φ differ by a factor $1/\sqrt{2\pi}$.

Finally, the width of $\varphi(k)$ should be obvious, it is a box of width $2K$. The width of $\psi(x)$ is a bit trickier. If you plot $\psi(x)$, you'll notice that most of the intensity comes in the region between the first two zeros, so the distance between the zeros would be a reasonable estimate of the "width." In fact, it is the same function that gives the intensity profile in problem 3, and this is no accident: the momentum wave function represents a slit filtering out all momenta except those within a narrow slit, so the position wave function looks like single-slit diffraction! Anyway: we know that the zeros of $\psi(x)$ will come when $\sin Kx=0$, or first at $Kx=\pm\pi$. This gives a width of $\Delta x=2\pi/K$, and

$$\Delta x\, \Delta k = \left(\frac{2\pi}{K}\right)(2K) = 4\pi \qquad \text{or} \qquad \Delta x\, \Delta p = 2h \tag{45}$$

This amounts to proving our previous results from problems 3 and 7: single slit diffraction obeys the uncertainty principle.

This problem was meant to be a mathematical warm-up to what we'll cover in more detail in the next few lectures – based on what you knew at the time, it was nothing more than a math problem, but hopefully its hidden purpose will become more clear shortly ...