University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair Fall 2010

Problem Set 6: Wave functions, 1D potentials

Instructions:

1. Answer all questions below. Show your work for full credit.
2. All problems are due Wed 6 October 2010 by the end of the day.
3. You may collaborate, but everyone must turn in their own work.

1. We found the wave functions and energies for a particle in an infinite potential well of width $2a$ to be

$$\psi^+(x) = \frac{1}{\sqrt{a}} \sin\left(\frac{n\pi x}{a}\right) \qquad E_n^+ = \frac{\left(n-\frac{1}{2}\right)^2 \pi^2\hbar^2}{2ma^2} \tag{1}$$

$$\psi^-(x) = \frac{1}{\sqrt{a}} \cos\left(\frac{\left(n-\frac{1}{2}\right)\pi x}{a}\right) \qquad E_n^- = \frac{n^2\pi^2\hbar^2}{2ma^2} \tag{2}$$

The $+$ and $-$ here indicate the even/odd solutions (i.e., an even or odd number of half-wavelengths fitting inside the well). Noting that $\langle p^2\rangle = 2mE_n^{\pm}$, calculate $\Delta p\,\Delta x$ for even and odd solutions, with $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$. How does the uncertainty behave for increasing n?

2. The state of a free particle is described by the following wave function

$$\psi(x) = \begin{cases} 0 & x < -b \\ A & -b \leqslant x \leqslant 2b \\ 0 & x > 2b \end{cases} \tag{3}$$

(a) Determine the normalization constant A.
(b) What is the probability of finding the particle in the interval $[0, b]$?
(c) Determine $\langle x\rangle$ and $\langle x^2\rangle$ for this state.
(d) Find the uncertainty in position $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$.

3. A particle of mass m is confined to a one-dimensional box of width L, that is, the potential energy of the particle is infinite everywhere except in the interval $0 < x < L$, where its potential energy is zero. The particle is in its ground state. What is the probability that a measurement of the particle's position will yield a result in the left quarter of the box? The wavefunction for a particle in a 1D box may be written

$$\psi(x) = A \sin(Bnx) \tag{4}$$

where A and B are constants you will need to find, and n is an integer. *Hint: normalize and apply boundary conditions.*

4. Given the wave function

$$\psi(x) = \frac{N}{x^2 + a^2} \tag{5}$$

(a) Find N needed to normalize ψ.
(b) Find $\langle x \rangle$, $\langle x^2 \rangle$, and Δx.
(c) What is the probability that the particle is found in the interval $[-a, a]$?

5. In electromagnetic theory, the conservation of charge is represented by the continuity equation (in one dimension)

$$\frac{\partial \vec{j}}{\partial x} = -e\frac{\partial \rho}{\partial t} \tag{6}$$

where $\vec{j}$ is current density and ρ charge density.

Identifying $|\psi(x)|^2$ as a 'probability density,' the quantum-mechanical analog of current density is

$$j(x) = -\frac{i\hbar e}{2m}\left(\psi^*\frac{\partial \psi}{\partial x} - \psi\frac{\partial \psi^*}{\partial x}\right) \tag{7}$$

(a) Show that the continuity equation above is satisfied with this definition of current density.
(b) For a bound-state wave function (a wave that isn't traveling), ψ can be chosen to be perfectly real, and $\psi^* = \psi$. What does this imply about the current density for bound states?
(c) Verify that the wave function from problem 4 gives zero current density everywhere.

6. A particle is in a stationary state in the potential $V(x)$. The potential function is now increased over all x by a constant value V_o. What is the effect on the quantized energy? Show that the spatial wave function of the particle remains unchanged.