University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair Fall 2010

Problem Set 8: Solutions

1. Transitions occur in an atom between an $l = 3$ and an $l = 2$ state in a field of $B = 0.2\,\mathrm{T}$. If the wavelength before the field was turned on was 400 nm, determine the final wavelengths observed.

Solution: $400 \pm 0.00149, 400\,\mathrm{nm}$

2. Calculate the frequency at which an electron's orbital magnetic moment μ precesses in a magnetic field B. Hint: open your PH106 book and look at the torque on a magnetic moment. This phenomenon is known as *Larmor precession.*

Solution: $eB/2m$
see `http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/larmor.html#c1`

3. For $l = 3$, calculate the possible angles $\vec{L}$ makes with the z axis.

Solution:

$$|\sin\theta| = \frac{L_z}{|\vec{L}|} = \frac{m_l}{\sqrt{l(l+1)}} = \frac{\{3, 2, 1\}}{\sqrt{3 \times 4}} = \{16.8^\circ, 35.3^\circ, 60^\circ\} \tag{1}$$

4. **(a)** What is the energy difference (in eV) between two electron spin orientations when the electrons are in a magnetic field of 0.5 T? **(b)** What wavelength of radiation could cause the electrons to "flip" their spins?

Solution: $58\,\mu\mathrm{eV}$, 2.14 cm

5. Estimate the strength of the magnetic field produced by the electron's orbital motion which results in the two sodium D lines (588.995 nm and 589.592 nm).

Solution: This is an example problem in your textbook. The transition occurs between an $L = 1$ state and an $L = 0$ state; only the $L = 1$ state will split due to the spin-orbit interaction. The difference in energy between the $L = 1$, $m_s = \frac{1}{2}$ and $L = 1$, $m_s = -\frac{1}{2}$ states will be $\Delta E = 2\mu_B B$ in an effective magnetic field B. The energy difference can also be equated to the wavelength difference, using the relationship given in problem 8:

$$\Delta E = E(m_s = \frac{1}{2}) - E(m_s = -\frac{1}{2}) \approx \frac{hc|\Delta\lambda|}{\lambda^2} = 2\mu_B B \tag{2}$$

Solving for B, with $\Delta\lambda = 589.592 - 588.995 \approx 0.597\,\mathrm{nm}$ and $\lambda = (589.592 + 588.995)/2 \approx 589.294\,\mathrm{nm}$,

$$B = \frac{hc\Delta\lambda}{2\mu_B \lambda^2} = \frac{(1240\,\mathrm{eV}\cdot\mathrm{nm})\,(0.597\,\mathrm{nm})}{2\,(57.9\times10^{-6}\,\mathrm{eV/T})\,(589.592\,\mathrm{nm})^2} \approx 18.4\,\mathrm{T} \tag{3}$$

6. Neglecting spin, in a strong external magnetic field of 5 T, determine the lines resulting from the $2p \to 1s$ transition ($\lambda_o = 121.0\,\mathrm{nm}$) in hydrogen. Provide a sketch of the energy levels and their m_l values.

Solution: The 2p level has $l = \{-1, 0, 1\}$, and the levels of different l will experience a Zeeman splitting and shift their energies by $l\mu_B B$. Thus, what is a single energy level in zero magnetic field becomes three distinct levels in a non-zero magnetic field, with energies

$$E_o, E_o \pm \mu_B B \tag{4}$$

where $E_o = hc/\lambda_o$. The 1s level has only $l = 0$, and thus experiences no Zeeman splitting. The new transitions thus have energies $E_o, E_o \pm \mu_B B$ rather than just E_o, so the new wavelengths are

$$\lambda = \left\{\lambda_o, \frac{hc}{E_o - \mu_B B}, \frac{hc}{E_o + \mu_B B}\right\} \approx \{121.0, 121.003, 120.997\}\ \mathrm{nm} \tag{5}$$

7. *Multiplicity of atomic magnetic moments.* Calculate the magnetic moments that are possible for the $n = 3$ level of Hydrogen, making use of the quantization of angular momentum. You may neglect the existence of spin. Compare this with the Bohr prediction for $n = 3$.

Solution: The magnetic moment can be related to the total orbital angular momentum:

$$|\vec{\mu}| = -\frac{e}{2m_e}|\vec{L}| = -\mu_B \frac{|\vec{L}|}{\hbar} \tag{6}$$

In turn, we know how the magnitude of $\vec{L}$ depends on l:

$$|\vec{L}| = \sqrt{l\,(l+1)}\hbar \tag{7}$$

So in general the moments will be

$$|\vec{\mu}| = -\mu_B \sqrt{l\,(l+1)} \tag{8}$$

For $n = 3$, we may have $l = \{0, 1, 2\}$, so this gives $|\vec{\mu}| = \left\{0, -\sqrt{2}, -\sqrt{6}\right\}\mu_B$.

In the Bohr model, our general relationship between $\vec{L}$ and $\vec{\mu}$ remains valid, but angular momentum is not given by a separate quantum number (the Bohr model has only the principle quantum number n), but simply by $L = n\hbar$. Thus, the moment for a given n in the Bohr model is single-valued, and given by

$$|\vec{\mu}_{\text{Bohr}}| = -\mu_B \frac{|\vec{L}|}{\hbar} = -n\mu_B \tag{9}$$

The Bohr model is in sharp disagreement with the full quantum solution.

8. *Transitions in a magnetic field.* Transitions occur in an atom between $l=2$ and $l=1$ states in a magnetic field of 0.6 T, obeying the selection rules $\Delta m_l = 0, \pm 1$. If the wavelength before the field was turned on was 500.0 nm, determine the wavelengths that are observed. You may find the following relationship useful:

$$\left|\Delta\lambda\right| = \frac{\lambda^2 \Delta E}{hc} \tag{10}$$

Recall that the Zeeman effect changes the energy of a single-electron atom in a magnetic field by

$$\Delta E = m_l \left(\frac{e\hbar}{2m_e}\right) B \qquad \text{with} \qquad m_l = -l, -(l-1), \ldots, 0, \ldots, l-1, l \tag{11}$$

For convenience, note that $e\hbar/2m_e = \mu_B \approx 57.9\,\mu\text{eV/T}$, and neglect the existence of spin.

Solution: In a magnetic field B, the energy levels for a given l state will split according to their value of m_l. If the original energy of the level is E_l, then the original level will be split symmetrically into $2l+1$ sub-levels, with adjacent levels shifted by $\mu_B B$:

$$E_{l,m_l} = E_l + m_l \mu_B B \tag{12}$$

This is shown schematically below for $l=2$ and $l=1$ levels. The $l=2$ level has possible m_l values of $m_l = \{-2, -1, 0, 1, 2\}$, and thus in a magnetic field B what was a single level is now 5 individual levels. For $l=1$, we have m_l values of only $m_l = \{-1, 0, 1\}$, and the original level becomes a triplet upon applying a magnetic field.

Before calculating anything, we can apply the dipole selection rules, which states that m_l can change by only $0, \pm 1$. This means that, for example, from the $l=2$, $m_l=1$ level an electron may "jump" to the any of the $l=1$, $m_l = \{2, 1, 0\}$ levels. On the other hand, from $l=2$, $m_l=2$ level an electron may only jump to the $l=1$, $m_l=1$ level. Following these rules, we see from the figure above that there are only 9 possible transitions allowed. Further, noting that the levels are equally spaced, we have in fact only three different transition energies.

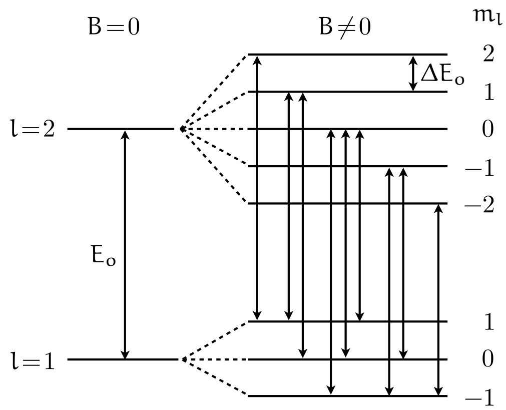


Figure 1: *Allowed transitions from* $l=2$ *to* $l=1$ *with a magnetic field applied.*

The spacing between the levels ΔE_o is the Zeeman energy given above, $\Delta E_o = \mu_B B$. From our schematic above, it is clear that the only possible transition energies in a magnetic field are the original transition energy (no change in m_l), or the original transition energy plus or minus ΔE_o (m_l changes by ± 1). The original transition energy E_o is readily found from the given wavelength $\lambda = 500\,\text{nm}$:

$$E_o = \frac{hc}{\lambda} \approx 2.5\,\text{eV} \tag{13}$$

Thus, the new transition energies must be

$$E_o \longmapsto \{E_o - \Delta E_o, E_o, E_o + \Delta E_o\} = \{E_o - \mu_B B, E_o, E_o + \mu_B B\} \tag{14}$$

That is, the original transition energy plus two new ones. We can easily convert these two new energies into two new wavelengths by the energy-wavelength relationship $E = hc/\lambda$. However, this does require some numerical precision (i.e., carrying at least 7-8 digits in your calculations, and knowing the requisite constants to commensurate precision), and it is somewhat easier to simply calculate the change in energy by itself. Since we know the energy changes by $\pm\Delta E_o$, using the formula given we have

$$\left|\Delta\lambda\right| = \frac{\lambda^2 \Delta E_o}{hc} = \frac{\lambda^2 \mu_B B}{hc} \approx 0.007\,\text{nm} \tag{15}$$

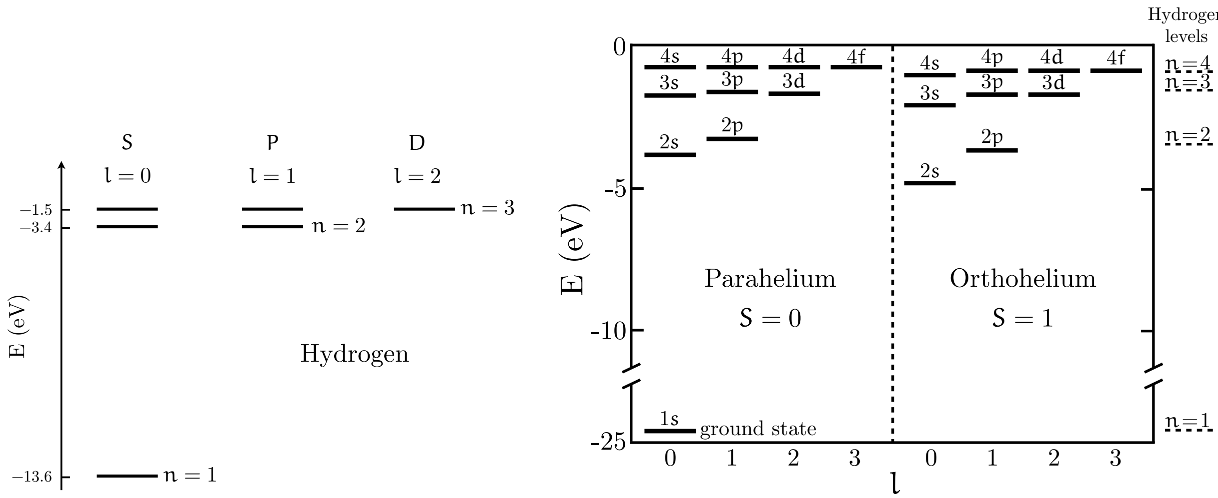
The shift in energy of ΔE_o implies a shift in wavelength of $\Delta\lambda \approx 0.007\,\text{nm}$, meaning the new transitions must be at the original wavelength λ plus or minus $\Delta\lambda$:

$$\lambda \longmapsto \{\lambda - \Delta\lambda, \lambda, \lambda + \Delta\lambda\} = \{499.994, 500.000, 500.007\}\ \mathrm{nm} \tag{16}$$

9. *Dipole selection rules.*

(a) For hydrogen, the energy levels through $n=3$ are shown below. What are the possible electric dipole transitions for these states? It may be convenient to simply draw arrows in the diagram. Recall the "selection rules" for electric dipole transitions, $\Delta l = \pm 1$. Spin may be ignored.

(b) Repeat for para- and ortho-helium, also shown below, treating both as distinct atoms.[i]



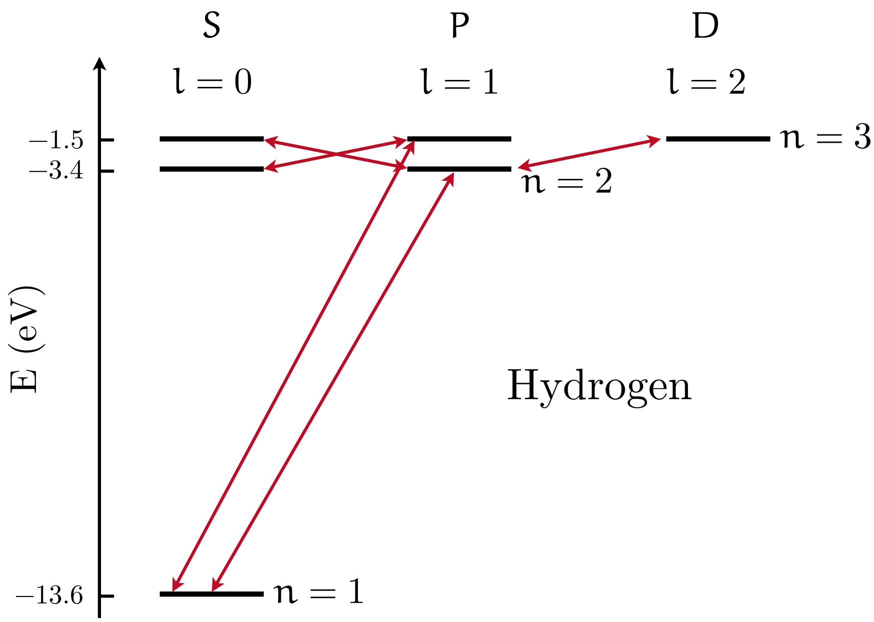
Problem 9. **(left)** *Energy levels of H through $n=3$, neglecting spin.* **(right)** *Energy levels of para- and ortho- He through $n=4$.*

Solution: You can change as many levels at a time as you like (i.e., move up or down arbitrarily), but the $\Delta l = \pm 1$ selection rule means you can must move one (and only one) unit to the left or right to make a transition. Thus, the problem is reduced to drawing lines in the figures given. Note that with the energy scales given, not all of the transitions are in the visible range – the longest lines will be in the ultraviolet, the shortest in the infrared (or microwave, for closely-spaced He levels).

[i] Two types of helium: para-helium, with the two electron spins parallel ($S=0$), and ortho-helium, with the two electron spins antiparallel ($S=1$). According to the dipole selection rules, helium atoms cannot change by a radiative process from one to the other, as this would not conserve angular momentum, so ortho- and para-helium behave largely as distinct atoms. (Forbidden transitions are not strictly forbidden, but violating the selection rules incurs a cost of $\sim 10^5$ in transition probability).

As the energy level diagram shows, the lowest state corresponds to para-helium, and the next highest excited state ortho-helium. The ortho-helium excited state can be reached by electrical discharge excitation, a non-radiative process (i.e., not obeying the same selection rules.) This excited state is very long lived ($\sim 10\,\mathrm{ms}$) because returning to the ground state would violate selection rules.

For the very closely-spaced levels (e.g., 3p and 3d) the transitions for He have been omitted for clarity; the diagram quickly becomes quite complicated!



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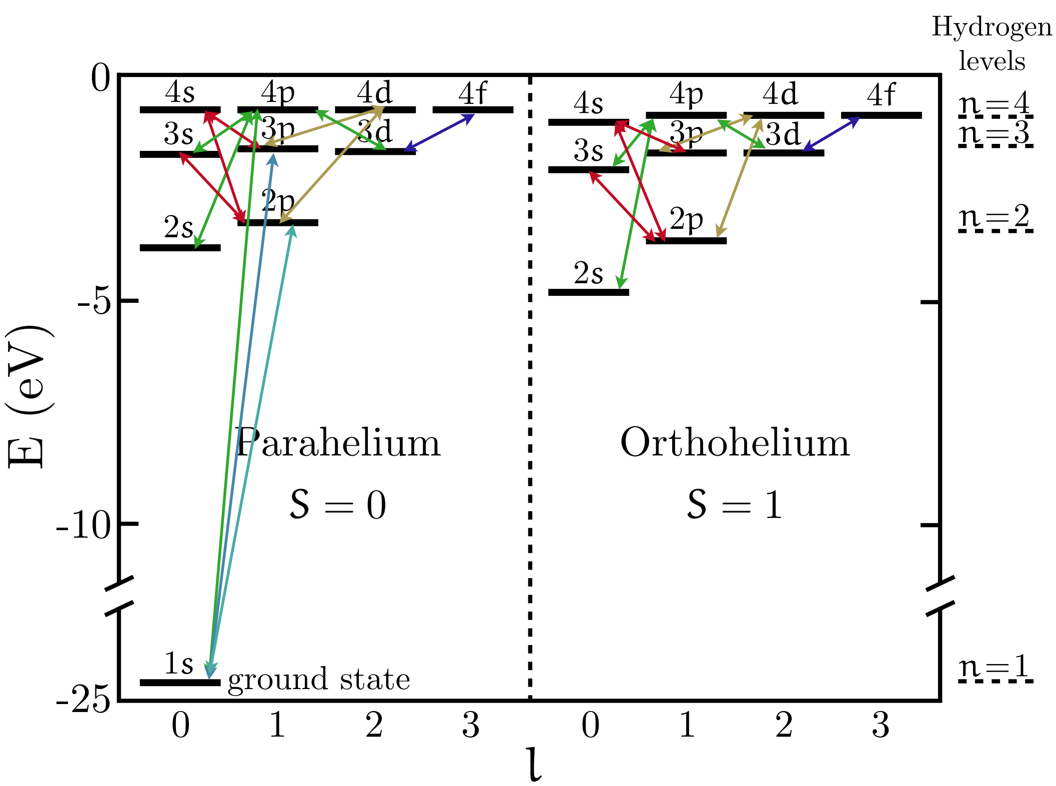

Hydrogen levels
0
-5
-10
-25
E (eV)
4s
3s
2s
1s
ground state
4p
3p
2p
4d
3d
4f
Parahelium
S = 0
Orthohelium
S = 1
n=4
n=3
n=2
n=1
0
1
2
3
l