# Problem Set 1: Solutions

**1. Daily problem due 23 Aug 2013:** How fast must a rocket travel relative to the earth so that time in the rocket "slows down" to half its rate as measured by earth-based observers? Do present-day jet planes approach such speeds?

***Solution:*** The question wants to know how much time is slowed down compared to the earth-based observers, which means the earth-based observers have the 'proper' time. The rocket must experience dilated time by comparison. Thus, the elapsed time must be related by

$$\Delta t'_{\text{rocket}} = \gamma \Delta t_{\text{earth}} = \frac{\Delta t_{\text{earth}}}{\sqrt{1 - v^2/c^2}} \tag{1}$$

Time slowing down by a factor two implies

$$\frac{\Delta t'_{\text{rocket}}}{\Delta t_{\text{earth}}} = 2 = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{2}$$

$$\Longrightarrow \frac{1}{4} = 1 - \frac{v^2}{c^2} \tag{3}$$

$$\Longrightarrow v = \frac{\sqrt{3}}{2} c \approx 0.866c \approx 2.6 \times 10^8 \text{ m/s} \tag{4}$$

**2. Daily problem due 26 Aug 2013:** A cube of metal with sides of length $a$ sits at rest in frame $S$ with one edge parallel to the $x$-axis. Therefore, in $S$ the cube has volume $a^3$. Frame $S'$ moves along the $x$-axis with speed $u$. As measured by an observer in frame $S'$, what is the volume of the metal cube?

***Solution:*** Since relative motion is involved, there must be length contraction for the moving observer - the person in $S'$ since we are observing the block. Since there is relative motion only along the $x$ axis, there is length contraction only along that axis. The cube therefore appears shortened by a factor $\gamma$ along the $x$ axis, but its dimensions along $y$ and $z$ are the same. The volume in the two frames is thus:

$$V = a \cdot a \cdot a = a^3 \qquad \text{in } S \tag{5}$$

$$V' = \frac{a}{\gamma} \cdot a \cdot a = \frac{a^3}{\gamma} \qquad \text{in } S' \tag{6}$$

***The problems below are due by the end of the day on 28 Aug 2013.***

**3.** One of the wavelengths of light emitted by hydrogen atoms under normal laboratory conditions is $\lambda = 656.3\,\text{nm}$, in the red portion of the electromagnetic spectrum. In the light emitted from a distant galaxy this same spectral line is observed to be Doppler-shifted to $\lambda = 953.4\,\text{nm}$, in the infrared portion of the spectrum. How fast are the emitting electrons moving relative to the earth? Are they approaching the earth or receding from it?

***Solution:*** The relativistic Doppler shift is given by

$$\lambda_o = \lambda_s \sqrt{\frac{c+v}{c-v}} \tag{7}$$

where $\lambda_o$ is the wavelength observed in relative motion at velocity $v$ with respect to the source, and $\lambda_s$ its he wavelength observed in the source's frame. Positive velocities correspond to observers approaching the source. We are given both wavelengths: $\lambda_o = 656.3\,\text{nm}$ and $\lambda_s = 953.4\,\text{nm}$. Since $\lambda_o > \lambda_s$, from the equation above it is clear that we must have $v < 0$ for this to be true, which already tells the source is receding. All we need to do is solve the above for $v$ to find the speed, which gives

$$\frac{v}{c} = \frac{\left(\frac{\lambda_o}{\lambda_s}\right)^2 - 1}{\left(\frac{\lambda_o}{\lambda_s}\right)^2 + 1} \approx 0.357 \tag{8}$$

**4.** Two particles in a high-energy accelerator experiment approach each other head-on with a relative speed of $0.890c$. Both particles travel at the same speed as measured in the laboratory. What is the magnitude of the velocity of one particle relative to the other?

***Solution:*** What we are given is the speed of the two particles relative to each other. That is, if we were in the reference frame of one of the particles, we would say the other approaches with $u{=}0.890c$. In the observer's frame (call it $S'$), we see the particles moving toward each other, each with the same speed $v'$. In the $S'$ frame of reference, we would have to say that adding the two velocities $v'$ together gives us $u{=}0.890c$.

$$u = \frac{v' + v'}{1 + v'v'/c^2} = 0.890c \tag{9}$$

Solving this for $v'$ gives us velocity of one particle relative to the other in the lab frame. The result is

$$v' = \frac{2 \pm \sqrt{4 - 4u^2/c^2}}{2u/c^2} = \frac{c^2}{u}\left(1 \pm \sqrt{1 - u^2/c^2}\right) \approx \{0.611c, 1.64c\} \tag{10}$$

Clearly, the second root, while mathematically allowed, does not make physical sense - we've already established velocities can't be greater than $c$. Therefore we reject it as unphysical, and the remaining valid solution is $v \approx 0.611c$.

**5.** **(a)** Through what potential difference does an electron have to be accelerated, starting from rest, to achieve a speed of $0.980c$? **(b)** What is the kinetic energy of the electron at this speed? Express your answer in both joules and electron volts.

***Solution:*** The key is to remember that a charge $q$ moving through a potential difference $\Delta V$ changes its potential energy by $\Delta U = q\Delta V$. If we are not worrying about resistive forces, this change in potential energy is equal to the charge's change in kinetic energy. Starting from rest, we know that must be $\Delta K = (\gamma - 1)\,mc^2$, with $\gamma = 1/\sqrt{1 - v^2/c^2}$.

For an electron $q = e$, and $mc^2 = 511\,\text{keV}$, and with $v = 0.980c$ we have $\gamma \approx 5.025$. Putting it all together,

$$\Delta K = \Delta U = e\Delta V = (\gamma - 1)\,mc^2 = (5.025 - 1)\left(511 \times 10^3\,\text{eV}\right) \approx 2.06 \times 10^6\,\text{eV} \tag{11}$$

Since $e\Delta V$ is the particle's change in both potential and kinetic energy, this is already the answer to the second part of the question: the particle's kinetic energy is about 2.06 MeV, or about 0.33 pJ ($\text{p} = 10^{-12}$). The corresponding potential difference is by definition 2.06 MV, illustrating how handy a unit the electron volt is.

**6.** Use the following two equations:

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1 - v^2/c^2}} \tag{12}$$

$$E = \frac{mc^2}{\sqrt{1 - v^2/c^2}} \tag{13}$$

to derive the following relationship:

$$E = \sqrt{(pc)^2 + (mc^2)^2}$$

***Solution:*** No big trick, just grind through it. Since you know the result you want to get to, start by finding $p^2c^2$.

$$p^2c^2 = \frac{m^2v^2c^2}{1 - v^2/c^2} = m^2c^4\left(\frac{v^2}{c^2 - v^2}\right) \tag{14}$$

Now add $m^2c^4$ and rearrange, and you've got it.

$$p^2c^2 + m^2c^4 = m^2c^4\left(1 + \frac{v^2}{c^2 - v^2}\right) = m^2c^4\left(\frac{c^2 + v^2 - v^2}{c^2 - v^2}\right) \tag{15}$$

$$= m^2c^4\frac{c^2}{c^2 - v^2} = m^2c^4\left(\frac{1}{1 - v^2/c^2}\right) = \gamma^2 m^2c^4 \tag{16}$$

$$\sqrt{p^2c^2 + m^2c^4} = \gamma mc^2 = E \tag{17}$$

**7.** A charge $q$ at $x=0$ accelerates from rest in a uniform electric field $\vec{\mathbf{E}}$ which is directed along the positive $x$ axis.

**(a)** Show that the acceleration of the charge is given by

$$a = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

**(b)** Show that the velocity of the charge at any time $t$ is given by

$$v = \frac{qEt/m}{\sqrt{1 + (qEt/mc)^2}}$$

**(c)** Find the distance the charge moves in a time $t$. *Hint:* ***http://integrals.wolfram.com***

***Solution:*** We are to find the acceleration, velocity, and position as a function of time for a particle in a uniform electric field. We are given the electric force and the boundary conditions $x=0$, $v=0$ at $t=0$. We will need only $F=dp/dt$, $p=\gamma mv$, the definition of $\gamma$ (given in previous problems), and a good knowledge of calculus (including the chain rule once again).

First, we must relate force and acceleration relativistically. Since velocity is explicitly a function of time here, so is $\gamma$, and we must take care.

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma mv) = \gamma m\frac{dv}{dt} + mv\frac{d\gamma}{dt} \tag{18}$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1 - v^2c^2}} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{2v}{c^2}\right)\frac{dv}{dt}}{(1 - v^2/c^2)^{3/2}} = \frac{v}{c^2}\frac{1}{(1 - v^2/c^2)^{3/2}} \tag{19}$$

$$F = \gamma m\frac{dv}{dt} + \frac{mv^2}{c^2}\frac{1}{(1 - v^2/c^2)^{3/2}}\frac{dv}{dt} = m\left(\frac{dv}{dt}\right)\left(\frac{1}{\sqrt{1 - v^2/c^2}} + \frac{\frac{v^2}{c^2}}{(1 - v^2/c^2)^{3/2}}\right) \tag{20}$$

$$F = ma\left(\frac{1 - \frac{v^2}{c^2}}{(1 - v^2/c^2)^{3/2}} + \frac{\frac{v^2}{c^2}}{(1 - v^2/c^2)^{3/2}}\right) = \frac{ma}{(1 - v^2/c^2)^{3/2}} \tag{21}$$

If you decided not to use $\gamma$ and wrote everything explicitly in terms of $1/\sqrt{1 - v^2/c^2}$, that is fine.

The end result is the same.

That accomplished, we can set the net force equal to the electric force $qE$ and solve for acceleration:

$$F = qE = \frac{ma}{(1 - v^2/c^2)^{3/2}} \tag{22}$$

$$a = \frac{qE}{m}\left(1 - v^2/c^2\right)^{3/2} \tag{23}$$

We can find velocity by writing $a$ as $dv/dt$ (as it was above) and noticing that the resulting equation is separable.

$$a = \frac{dv}{dt} = \frac{qE}{m}\left(1 - v^2/c^2\right)^{3/2} \tag{24}$$

$$\frac{qE}{m}\,dt = \frac{dv}{(1 - v^2/c^2)^{3/2}} \tag{25}$$

We can now integrate both sides, noting from the boundary conditions that if time runs from 0 to $t$, the velocity runs from 0 to $v$.

$$\int_0^v \frac{dv}{(1 - v^2/c^2)^{3/2}} = \int_0^t \frac{qE}{m}\,dt \tag{26}$$

$$\left.\frac{v}{\sqrt{1 - v^2/c^2}}\right|_0^v = \left.\frac{qEt}{m}\right|_0^t \tag{27}$$

$$\frac{v}{\sqrt{1 - v^2/c^2}} = \frac{qEt}{m} \tag{28}$$

Solving for $v$, we first square both sides ...

$$\frac{v^2}{1 - v^2/c^2} = \frac{q^2E^2t^2}{m^2} \tag{29}$$

$$v^2 = \left(1 - \frac{v^2}{c^2}\right)\frac{q^2E^2t^2}{m^2} \tag{30}$$

$$v^2\left(1 + \frac{q^2E^2t^2}{m^2c^2}\right) = \frac{q^2E^2t^2}{m^2} \tag{31}$$

$$v = \frac{qEt/m}{\sqrt{1 + (qEt/mc)^2}} \tag{32}$$

We can find position by integrating $v$ through time from 0 to $t$, which is straightforward.

$$x = \int_0^t \frac{qEt/m}{\sqrt{1+(qEt/mc)^2}}\,dt = \frac{mc^2}{qE}\sqrt{1+\left(\frac{qEt}{mc}\right)^2}\,\Bigg|_0^t = \frac{mc^2}{qE}\left(\sqrt{1+\left(\frac{qEt}{mc}\right)^2}-1\right) \tag{33}$$

Classically, we would expect a parabolic path, but in relativity we find the path is a *hyperbola*.[i] Also note that the position, velocity, and acceleration depend overall on the ratio between the particle's rest energy $mc^2$ to the electric force $qE$ (note energy/force is distance).

**8.** Show that for the preceding question the particle's speed approaches $c$ as $t \to \infty$.

***Solution:*** This is basically a way of double-checking that our previous result makes sense. It also reinforces the idea that even with a constant, steady acceleration for infinite time nothing is going to reach the speed of light.

All we need to do is take the $t \to \infty$ limit of our velocity expression. Divide everything by $t$ and it is straightforward.

$$\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{qEt/m}{\sqrt{1+(qEt/mc)^2}} = \lim_{t\to\infty} \frac{qE/m}{\sqrt{1/t^2+(qE/mc)^2}} = \frac{qE/m}{qE/mc} = c \tag{34}$$

**9.** At what speed is the momentum of a particle twice as great as the result obtained from the non-relativistic expression $mv$? Express your answer in terms of the speed of light.

***Solution:*** Relativistic momentum is $p = \gamma mv$, classically we would write $p = mv$. The latter is off by a factor of two when

$$\gamma mv = 2mv \tag{35}$$

$$\gamma = 2 \tag{36}$$

$$|v| = \frac{\sqrt{3}}{2}c \tag{37}$$

**10.** Light travels with respect to earth at $3 \times 10^8\ \frac{m}{s}$. A rocket travels at $2.5 \times 10^8\ \frac{m}{s}$ with respect to earth in opposite direction of the light. What is the speed of light as viewed from the rocket?

***Solution:*** It is light, the speed is always $c$ in vacuum.

[i] If you square both sides, the equation for $x(t)$ can be put in the form $x^2/a^2 - t^2/b^2 = 1$, the standard form of the equation for a hyperbola.