Problem Set 2: Solutions

Daily problem due 4 Sept 2013: Recall the formula we developed for the electric field of a charge in motion at constant velocity. Calculate the field strength using that expression for the limiting cases of **(a)**: $\theta\!=\!0$, **(b)** $\theta\!=\!90°$, **(c)** $v\!=\!0$.

Solution: Our expression for the field of a charge moving at constant velocity was:

$$\frac{q}{4\pi\epsilon_o r^2}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}} \tag{1}$$

where θ is the angle with respect to the axis of motion, which is also the x axis. Along the horizontal axis ($\theta=0°$), the field is reduced by a factor γ^2 compared to what it would be for a stationary charge,

$$E' = \frac{kq}{\gamma^2 r'^2} \qquad \text{(along } x'\text{)} \tag{2}$$

while along the vertical axis ($\theta\!=\!90°$), the field is *enhanced* by a factor γ:

$$E' = \frac{kq\gamma}{r'^2} \qquad \text{(along } z'\text{)} \tag{3}$$

Daily problem due 6 Sept 2013: Which of the following expressions correspond to traveling waves? For each of those, what is the speed of the wave? The quantities A, a, b, c are positive real constants.

$$\psi(x,t) = (ax-bt)^2 \tag{4}$$
$$\psi(x,t) = A\sin\left(ax^2-bt^2\right) \tag{5}$$
$$\psi(x,t) = \frac{1}{ax^2+b} \tag{6}$$
$$\psi(x,t) = A\sin 2\pi\left(\frac{x}{a}+\frac{t}{b}\right) \tag{7}$$

Solution: In order to be a traveling wave, the wavefunction must take the form $f(\alpha x\!\pm\!\beta t)$. Thus, only the first and last functions are traveling waves. The velocity is the ratio of the coefficient of the time term to the spatial term, $v\!=\!-\beta/\alpha$, and the sign of the time term tells us the direction. Thus for the first the velocity is $v\!=\!b/a$ in the $+x$ direction, and for the last $v\!=\!a/b$ along $-x$

You say the argument of the second function can be factored into $\left(\sqrt{a}x - \sqrt{b}t\right)\left(\sqrt{a}x - \sqrt{b}t\right)$, and then maybe you could use some trigonometric identities to put it in the form $f(\alpha x \pm \beta t)$. Give it a try – turns out it doesn't work. Can you guess why?

The problems below are due by the end of the day on 9 Sept 2013.

1. **(a)** Charge q_a is at rest at the origin in system S; charge q_b flies by at speed v on a trajectory parallel to the x axis, but at $y = d$. What is the electromagnetic force on q_b as it crosses the y axis?
(b) Now study the same problem from system S', which moves to the right with speed v. What is the force on q_b when q_a passes the y' axis? You can either use your previous answer and transform the force, or compute the fields in S' using the Lorentz force law.

Solution: **(a)** The fields of charge a at charge b are just those of a static charge a distance d away. The force is then just

$$\vec{\mathbf{F}} = \frac{kq_a q_b}{d^2}\hat{\mathbf{y}} \tag{8}$$

(b) The field of charge a at charge b is now that of a charge moving at constant velocity v, sitting at a distance d away at an angle of $\theta = 90°$ with respect to the axis of motion. From the first problem we deduced that the electric field is *enhanced* by a factor γ in this case, making the force

$$\vec{\mathbf{F}} = \frac{\gamma kq_a q_b}{d^2}\hat{\mathbf{y}} \tag{9}$$

2. A proton is uniformly accelerated in a van de Graaff accelerator through a potential difference of 700 kV. The length of the linear accelerating region is 3 m. **(a)** Compute the ratio of the radiated energy to the final kinetic energy. **(b)** Show that for a particle moving in a linear accelerator the rate of radiation of energy is

$$\frac{dU}{dt} = \frac{q^2}{6\pi\epsilon_o m^2 c^3}\left(\frac{dK}{dx}\right)^2 \tag{10}$$

where K is the kinetic energy.

Solution: The energy radiated in a time t can be found from the Larmor formula for radiated power

$$U_{\text{rad}} = Pt = \frac{q^2 a^2 t}{6\pi\epsilon_o c^3} \tag{11}$$

The time it takes the proton to move through a distance s with acceleration a and initial velocity zero we can find from kinematics. The distance covered must be $s=\frac{1}{2}at^2$ and the velocity $v=at$, which we can combine to give $s=\frac{1}{2}vt$. That in turn implies $t=2s/v$ and $a=v/t=v^2/2s$. Inserting these results into our energy equation:

$$U_{\text{rad}} = \frac{q^3v^3}{12\pi\epsilon_o c^3 s} \tag{12}$$

This still leaves everything in terms of the velocity, which we can get from conservation of energy: the potential energy change due to the accelerating potential difference ΔV must be the same as the kinetic energy change.

$$K = \frac{1}{2}mv^2 = q\Delta V \qquad \Longrightarrow \qquad v = \sqrt{\frac{2q\Delta V}{m}} \tag{13}$$

where m is the proton mass. Putting it all together,

$$\frac{U_{\text{rad}}}{K} = \frac{1}{\frac{1}{2}mv^2}\frac{q^3v^3}{12\pi\epsilon_o c^3 s} = \frac{q^3 v}{6\pi\epsilon_o mc^3 s} = \frac{q^3}{6\pi\epsilon_o mc^3 s}\sqrt{\frac{2q\Delta V}{m}} \tag{14}$$

$$\frac{U_{\text{rad}}}{K} = \frac{q^3}{6\pi\epsilon_o c^3 s}\sqrt{\frac{2q\Delta V}{m^3}} \approx 1.31 \times 10^{-20} \tag{15}$$

Our conclusion is that radiation losses in linear accelerators are utterly negligible. The rate of energy loss can be related to the kinetic energy gained per unit distance (dK/dx):

$$K = \frac{1}{2}mv^2 \tag{16}$$

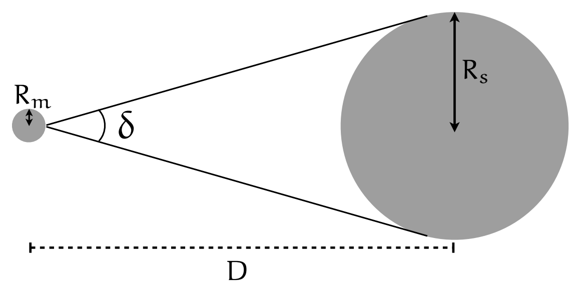
$$\frac{dK}{dx} = mv\frac{dv}{dx} = mv\frac{dv}{dt}\frac{dt}{dx} = ma \tag{17}$$

For the last step, the chain rule was employed to break up dv/dx into more manageable bits, and note that $dt/dx=1/v$. Given this, we can substitute $a=\frac{1}{m}\frac{dK}{dx}$ into our power equation and obtain the desired result:

$$\frac{dU}{dt} = \frac{q^2a^2}{6\pi\epsilon_o m^2c^3} = \frac{q^2}{6\pi\epsilon_o m^2c^3}\left(\frac{dK}{dx}\right)^2 \tag{18}$$

3. Assume the sun radiates like a black body at 5500 K. Assume the moon absorbs all the radiation it receives from the sun and reradiates an equal amount of energy like a black body at temperature T. The angular diameter of the sun seen from the moon is about 0.01 rad. What is the equilibrium temperature T of the moon's surface? (Note: you do not need any other data than what is contained in the statement above.

Solution: The geometry of the problem is shown below, where δ is the angular diameter, R_m the moon's radius, R_s the sun's radius, and D the sun-moon distance.



The definition of angular diameter[i], using the distances in the figure above, is

$$\tan\frac{\delta}{2} = \frac{R_s}{D} \tag{19}$$

Keep in mind that the link on the Wikipedia is using diameter instead of radius for some misguided reason.

With geometry in hand, we now need to balance the sun's power received by the moon with the power that the moon will re-radiate by virtue of its being at temperature T_m. Any body at temperature T emits a power $P = \sigma T^4 A$, where A is the area over which the radiation is emitted and σ is a constant. Thus, since the sun emits radiation over its whole surface area $4\pi R_s^2$,

$$P_s = \sigma T_s^4 \left(4\pi R_s^2\right) \tag{20}$$

At a distance D corresponding to the moon's position, this power is spread over a sphere of radius D and surface area $4\pi D^2$. The amount of power the moon receives just depends on the ratio its absorbing area to the total area over which the power is spread out. The moon absorbs radiation over an area corresponding to its cross section, πR_m^2, so the fraction of the sun's total power that the moon receives is $\pi R_m^2/4\pi D^2$. Thus, the moon receives a power

$$P_{mr} = P_s \frac{\pi R_m^2}{4\pi D^2} = P_s \frac{R_m^2}{4D^2} = \sigma T_s^4 \left(4\pi R_s^2\right) \frac{R_m^2}{4D^2} \tag{21}$$

Absorbing this radiation from the sun will cause the moon to heat up to temperature T_m, and it will re-emit radiation as a black body at temperature T_m. Though the moon absorbs over its

[i]See, e.g., `http://en.wikipedia.org/wiki/Angular_diameter`

cross-sectional area, it emits over its whole surface area, so its emitted power is

$$P_{me} = \sigma T_m^4 \left(4\pi R_m^2\right) \tag{22}$$

Equilibrium requires that the power the moon receives equal the power the moon emits, so

$$\begin{align} P_{mr} &= P_{me} \tag{23} \\ \sigma T_s^4 \left(4\pi R_s^2\right) \frac{R_m^2}{4D^2} &= \sigma T_m^4 \left(4\pi R_m^2\right) \tag{24} \\ T_s^4 \frac{R_s^2}{4D^2} &= T_m^4 \tag{25} \\ T_m = T_s \sqrt{\frac{R_s}{2D}} &= T_s \sqrt{\frac{1}{2} \tan \frac{\delta}{2}} \approx 275\,\mathrm{K} \tag{26} \end{align}$$

Compare this with a mean lunar surface temperature at the equator of 220 K – not bad given the approximate geometry, and complete ignorance of reflection! It is interesting to see that the moon's radius does not factor in at all – it determines both the absorbed and emitted power in exactly the same way, and ends up canceling out.

4. Presume the surface temperature of the sun to be 5500 K, and that it radiates approximately as a blackbody. What fraction of the sun's energy is radiated in the visible range of $\lambda = 400 - 700\,\mathrm{nm}$? One valid solution is to plot the energy density on graph paper and find the result numerically.

Solution: The emitted power per unit area per unit wavelength for a blackbody is given in a previous problem:

$$I(\lambda, T) = \frac{8\pi hc^2}{\lambda^5} \left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} \tag{27}$$

The power per unit area emitted over a range of wavelengths λ_1 to λ_2 is found by integrating $I(\lambda, T)$ over those limits, and the total power is integrating over all wavelengths from 0 to ∞. The fraction we desire is then the power over wavelengths λ_1 to λ_2 divided by the total power:

$$f = (\text{fraction}) = \frac{\int\limits_{\lambda_1}^{\lambda_2} I(\lambda, T)\, d\lambda}{\int\limits_0^{\infty} I(\lambda, T)\, d\lambda} \tag{28}$$

Let us first worry about the indefinite integral and put it in a bit simpler form.

$$\int I(\lambda, T)\, d\lambda = \int \frac{8\pi hc^2}{\lambda^5} \left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} d\lambda \tag{29}$$

It is convenient to make a change of variables to

$$u = \frac{hc}{\lambda k_b T} \qquad \text{or} \qquad \lambda = \frac{hc}{u k_b T} \tag{30}$$

This substitution implies

$$du = \frac{hc}{k_b T}\left(\frac{-d\lambda}{\lambda^2}\right) = -\frac{hc}{k_b T}\left(\frac{k_b T u}{hc}\right)^2 d\lambda = -\frac{u^2 k_b T}{hc} d\lambda \tag{31}$$

$$d\lambda = -\frac{hc}{u^2 k_b T} du \tag{32}$$

Performing the substitution,

$$\int I(\lambda, T)\, d\lambda = \int \frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} d\lambda = \int \frac{8\pi hc^2 u^5 k_b^5 T^5}{h^5 c^5} \frac{1}{e^u - 1} \frac{-hc}{u^2 k_b T} du \tag{33}$$

$$= -\frac{8\pi k_b^4 T^4}{h^3 c^2} \int \frac{u^3}{e^u - 1}\, du \tag{34}$$

The overall constants multiplying the integral will cancel in the fraction we wish to find:

$$f = \frac{\frac{8\pi k_b^4 T^4}{h^3 c^2} \int\limits_{u_1}^{u_2} \frac{u^3}{e^u - 1}\, du}{\frac{8\pi k_b^4 T^4}{h^3 c^2} \int\limits_{0}^{\infty} \frac{u^3}{e^u - 1}\, du} = \frac{\int\limits_{u_1}^{u_2} \frac{u^3}{e^u - 1}\, du}{\int\limits_{\infty}^{0} \frac{u^3}{e^u - 1}\, du} \tag{35}$$

Here the new limits of integration for the numerator are $u_1 = \frac{hc}{\lambda_1 k_b T} \approx 6.55\,\mathrm{m}^{-1}$ and $u_2 = \frac{hc}{\lambda_1 k_b T} \approx 3.74\,\mathrm{m}^{-1}$, and the denominator has limits of ∞ and 0 after the substitution.

$$f = \frac{\int\limits_{6.55}^{3.74} \frac{u^3}{e^u - 1}\, du}{\int\limits_{\infty}^{0} \frac{u^3}{e^u - 1}\, du} \tag{36}$$

As it turns out, the integral in the denominator is known, and has a numerical value of $\pi^4/15$. The integral in the numerator has no closed-form solution, and must be found numerically. One easy thing is to just go to `wolframalpha.com` and enter

integral of x^3/(e^x-1) dx from 3.74 to 6.55

And you'll find the answer is about 2.29. Given $\pi^4/15 \approx 6.49$, our ratio is $f \approx 35\%$.

Assuming for some odd reason you didn't have the internet at your disposal, all hope is not lost. One thing we notice is that the denominator contains a factor $e^u - 1$, and at the limits of integration we have

$$e^{3.74} \approx 42 \tag{37}$$
$$e^{6.55} \approx 700 \tag{38}$$

In this case, since $e^u \gg 1$, to a good approximation we can write

$$\frac{1}{e^u - 1} \approx \frac{1}{e^u} = e^{-u} \tag{39}$$

The error we make in this approximation is in the worst case of order $1/43 \sim 2\%$ This makes the integral in the numerator of our fraction a known one, which can be integrated by parts[ii]:

$$\int\limits_{6.55}^{3.74} \frac{u^3}{e^u - 1}\,du \approx \int\limits_{6.55}^{3.74} \frac{u^3}{e}^{-u}\,du = e^{-u}\left(u^3 + 3u^2 + 6u + 6\right)\Big|_{6.55}^{3.74} \approx 2.29 \tag{40}$$

Thus,

$$f \approx \frac{2.29}{\pi^4/15} \approx 0.35 \tag{41}$$

About 35% of the sun's radiation should be in the visible range.[iii] A more exact numerical calculation gives closer to 36%, meaning our approximation above was indeed accurate to about 2%.

5. An electron is released from rest and falls under the influence of gravity. **(a)** How much power does it radiate? **(b)** How much energy is lost after it falls 1 m? (*Hint:* $P = \Delta K/\Delta t$, $y = \frac{1}{2}gt^2$.)

Solution: The power emitted by a charge e with acceleration a is

$$P = \frac{e^2 a^2}{6\pi\epsilon_o c^3} \tag{42}$$

In this case, under free fall the electron's acceleration is $g \approx 9.81\,\mathrm{m/s^2}$, which gives

$$P = \frac{e^2 g^2}{6\pi\epsilon_o c^3} \approx 5 \times 10^{-52}\,\mathrm{W} \tag{43}$$

In a time t, starting from rest, an object under the influence of gravity falls a distance $\Delta y = \frac{1}{2}gt^2$.

[ii] Or with Wolfram ...

[iii] This is what *leaves the sun*, to figure out what reaches the earth's surface we would have to account for reflection and absorption by the atmosphere. The fraction of visible light is closer to 42% at the earth's surface; see `uvb.nrel.colostate.edu/UVB/publications/uvb_primer.pdf` for example.

Knowing the electron falls $\Delta y = 1\,\mathrm{m}$, the time it takes is

$$t = \sqrt{\frac{2\Delta y}{g}} \approx 0.45\,\mathrm{s} \tag{44}$$

Since the power dissipation is constant, the energy lost is just power times time (since $P = \Delta E/\Delta t$):

$$\Delta E = Pt = \frac{e^2 g^2}{6\pi\epsilon_o c^3}\sqrt{\frac{2\Delta y}{g}} \approx 2.5 \times 10^{-52}\,\mathrm{J} \tag{45}$$

An utterly negligible amount. We don't need to worry about radiation of charges accelerated by gravity.

6. An electron initially moving at constant speed v is brought to rest with uniform deceleration a lasting for a time $t = v/a$. Compare the electromagnetic energy radiated during this deceleration with the electron's initial kinetic energy. Express the ratio in terms of two lengths, the distance light travels in time t and the classical electron radius $r_e = e^2/4\pi\epsilon_o mc^2$.

Solution: The power emitted by a charge e with acceleration a is

$$P = \frac{e^2 a^2}{6\pi\epsilon_o c^3} \tag{46}$$

In this case, we know that $a = v/t$. The energy radiated in time t is just $U = Pt$, so

$$U = Pt = \frac{e^2 v^2}{6\pi\epsilon_o c^3 t} \tag{47}$$

The ratio of this energy to the kinetic energy before deceleration is

$$\frac{U}{K} = \frac{1}{\frac{1}{2}mv^2}\frac{e^2 v^2}{6\pi\epsilon_o c^3 t} = \frac{e^2}{3\pi\epsilon_o mc^3 t} \tag{48}$$

Noting that the distance light travels in a time t is $r_l = ct$ and using the expression for the classical electron radius above,

$$\frac{U}{K} = \frac{e^2}{3\pi\epsilon_o mc^3 t} = \frac{e^2}{4\pi\epsilon_o mc^2} \cdot \frac{4}{3} \cdot \frac{1}{ct} = \frac{4r_e}{3r_l} \tag{49}$$

7. A capacitor consists of two parallel rectangular plates with a vertical separation of 0.02 m. The east-west dimension of the plates is 0.2 m, the north-south dimension is 10 cm. The capacitor has been charged by connecting it temporarily to a battery of 300 V.

(a) How many excess electrons are on the negative plate?
(b) What is the electric field strength between the plates?

Now, give the quantities as they would be measured in a frame of reference which is moving eastward, relative to the laboratory in which the plates are at rest, with speed $0.6c$.

(c) The dimensions of the capacitor,
(d) The number of excess electrons on the negative plate,
(e) The electric field strength between the plates.

Solution: **(a)** The excess charge can be found from the definition of the capacitance and its specific form for two parallel plates:

$$C = \frac{Q}{\Delta V} = \frac{\epsilon_o A}{d} \qquad \Longrightarrow \qquad Q = \frac{\epsilon_o A \Delta V}{d} \approx 2.665^{-9}\,\mathrm{C} \approx 1.66 \times 10^{10}\,\text{electrons} \tag{50}$$

Here ΔV is the potential difference applied to the battery, the area of the plates is the product of the east-west and north-south distance, $l_{ew} l_{ns}$, and d is the vertical separation.

(b) The electric field strength between the plates, treating them as infinite plates, can be found in two ways:

$$E = \frac{\Delta V}{d} = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_o A} = 15,000\,\mathrm{V/m} \tag{51}$$

(c) Moving eastward, perpendicular to the direction separating the plates, we will have a contraction of the east-west length but *not* the north-south length or the separation. Thus, the new dimensions of the capacitor are

$$l'_{ew} = l_{ew}/\gamma = l_{ew}\sqrt{1 - v^2/c^2} = 0.16\,\mathrm{m} \tag{52}$$
$$l'_{ns} = l_{ns} = 0.1\,\mathrm{m} \tag{53}$$
$$d' = d = 0.02\,\mathrm{m} \tag{54}$$

(d) The number of electrons per plate is the same, since charge is invariant.

(e) The electric field strength will increase, since we have the same number of electrons confined to effectively smaller plates. The area of the plates is now a factor of gamma smaller, $l'_{ew} l'_{ns} = l_{ew} l_{ns}/\gamma$, meaning the charge density is a factor γ higher, and thus the electric field is also a factor of γ higher:

$$E' = \frac{\sigma'}{\epsilon_o} = \frac{\gamma\sigma}{\epsilon_o} = \gamma E = 18,750\,\mathrm{V/m} \tag{55}$$

When the motion of the capacitor is upward, only the distance between the plates is contracted:

$$l'_{ew} = l_{ew} = 0.2\,\mathrm{m} \tag{56}$$
$$l'_{ns} = l_{ns} = 0.1\,\mathrm{m} \tag{57}$$
$$d' = d/\gamma = 0.016\,\mathrm{m} \tag{58}$$

The number of electrons remains the same, as does the charge density in this case. The plates move closer together, but for an infinite parallel plate capacitor, the electric field *does not depend on the plate spacing*, $E = \sigma/\epsilon_o$. Thus, the electric field is unchanged when the motion is along the direction of the electric field.

8. *Hecht 2.38* Show that the imaginary part of a complex number z is given by

$$\frac{z - z^*}{2i} \tag{59}$$

Solution: Let $z = x + iy$ without loss of generality. Then $z^* = x - iy$, and

$$z - z^* = (x + iy) - (x - iy) = 2iy \tag{60}$$
$$\frac{z - z^*}{2i} = \frac{2iy}{2i} = y. \tag{61}$$

9. The equation for a driven damped oscillator is

$$\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{q}{m}E(t) \tag{62}$$

(a) Explain the significance of each term.
(b) Let $E = E_o e^{i\omega t}$ and $x = x_o e^{i(\omega t - \alpha)}$ where E_o and x_o are real quantities. Substitute into the above expression and show that

$$x_o = \frac{qE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \tag{63}$$

(c) Derive an expression for the phase lag α, and sketch it as a function of ω, indicating ω_o on the sketch.

Solution: The significance of each term is probably more apparent if we re-arrange and multiply by mass:

$$m\frac{d^2x}{dt^2} = -m\omega_o^2 x - 2\gamma m\omega_o\frac{dx}{dt} + qE(t) \tag{64}$$

The term on the right is the net force on the oscillator. The first term on the left is the restoring force, the second the viscous damping term, and the last the driving force of the oscillator.

First, we find the derivatives of x, noting $i^2 = -1$:

$$\frac{dx}{dt} = i\omega x_o e^{i(\omega t-\alpha)} \tag{65}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x_o e^{i(\omega t-\alpha)} \tag{66}$$

Substituting into the original equaiton,

$$\frac{q}{m}E_o e^{i\omega t} = -\omega^2 x_o e^{i(\omega t-\alpha)} + 2\gamma\omega_o i\omega x_o e^{i(\omega t-\alpha)} + \omega_o^2 x_o e^{i(\omega t-\alpha)} \tag{67}$$

$$\frac{q}{m}E_o e^{i\omega t} = e^{i(\omega t-\alpha)}\left(-\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o\right) \tag{68}$$

$$\frac{q}{m}E_o e^{i\omega t} = e^{i\omega t}e^{-i\alpha}\left(-\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o\right) \tag{69}$$

$$\frac{qE_o}{m}e^{i\alpha} = -\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o \tag{70}$$

To proceed, we use the Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{71}$$

Giving

$$\frac{qE_o}{m}\left(\cos\alpha + i\sin\alpha\right) = -\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o \tag{72}$$

We now actually have two distinct equations if we separately equate the purely real and purely imaginary parts (since they can't equal one other):

$$\frac{qE_o}{m}\cos\alpha = \omega_o^2 x_o - \omega^2 x_o \qquad \text{real terms} \tag{73}$$

$$\frac{qE_o}{m}\sin\alpha = 2\gamma\omega\omega_o x_o \qquad \text{imaginary terms} \tag{74}$$

We can square both equations and add them together:

$$\frac{q^2E_o^2}{m^2}\left(\cos^2\alpha+\sin^2\alpha\right)=\left(\omega_o^2-\omega^2\right)^2x_o^2+\left(2\gamma\omega\omega_o\right)^2x_o^2 \tag{75}$$

$$x_o^2=\frac{q^2E_o^2}{m^2}\frac{1}{\left(\omega_o^2-\omega^2\right)^2x_o^2+\left(2\gamma\omega\omega_o\right)^2} \tag{76}$$

$$x_o=\frac{qE_o}{m}\frac{1}{\sqrt{\left(\omega_o^2-\omega^2\right)^2x_o^2+\left(2\gamma\omega\omega_o\right)^2}} \tag{77}$$

This is the desired amplitude of vibration. Going back to the preceding two equations, we can divide the second equation by the first to find the phase angle:

$$\tan\alpha=\frac{2\gamma\omega\omega_o}{\omega_o^2-\omega^2}=\frac{2\gamma\left(\frac{\omega}{\omega_o}\right)}{1-\left(\frac{\omega}{\omega_o}\right)^2} \tag{78}$$

Using the last form, we can now easily plot α versus ω/ω_o. Picking modest damping factors of 0.1, 0.05, and 0.01, you can do something like this at `wolframalpha.com`

plot of y = arctan(2*0.1*x/(1-x^2)) and y = arctan(2*0.05*x/(1-x^2)) and y = arctan(2*0.01*x/(1-x^2))

you can also add things like “from x=0 to x=2” to change the plot range.