Problem Set 3 Solutions

Daily problem due 18 Sept 2013: Determine the maximum scattering angle in a Compton experiment for which the scattered photon can produce a positron-electron pair. *Hint:* twice the electron's rest energy m_ec^2 is required of the incident photon for pair production.

Solution: In order to produce a positron-electron pair, enough energy must be present to supply the rest energy of a positron and an electron. Both have the same mass, m_e, so that means the incident photon must supply at least $2m_ec^2$ worth of enery. The threshold wavelength is thus

$$hf = \frac{hc}{\lambda_{th}} = 2m_ec^2 \qquad \text{or} \qquad \frac{h}{m_ec} = 2\lambda_{th} \tag{1}$$

Any wavelengths *above* this value cannot result in pair production. Substituting this into the Compton formula,

$$\lambda_f = \lambda_i + \frac{h}{m_ec}\left(1-\cos\theta\right) = \lambda_i + 2\lambda_{th}\left(1-\cos\theta\right) \tag{2}$$

The right side of the expression is the sum of two positive-definite terms. The right side can be at most λ_{th} - if $\lambda_f > \lambda_{th}$, pair production cannot occur. Even if λ_i is arbitrarily small, if

$$2\lambda_{th}\left(1-\cos\theta\right) \geq \lambda_{th} \tag{3}$$

then $\lambda_f > \lambda_{th}$. This leads us to a condition on the threshod angle θ_{th}

$$1-\cos\theta_{th} = \frac{1}{2} \qquad \text{or} \qquad \theta_{th} = 60^\circ \tag{4}$$

Alternate Solution

All this means is that the exiting (scattered) photon must have an energy of at least $2mc^2$. In terms of the dimensionless photon energies $\alpha_i = hf_i/mc^2$, $\alpha_f = hf_f/mc^2$, the Compton equation reads

$$\frac{1}{\alpha_i} = \frac{1}{\alpha_f} - \left(1-\cos\theta\right) \tag{5}$$

If the exiting photon energy is $hf_f = 2mc^2$, this means $\alpha_f = 2$. Solving the Compton equation for α_i,

$$\alpha_i = \frac{1}{\frac{1}{\alpha_f} - \left(1-\cos\theta\right)} \tag{6}$$

Physically, α_i is an energy and it must be positive – that is the most basic requirement we can make. In the equation above, the numerator is clearly always positive, so the only condition we can enforce is that the denominator remain positive. This requires

$$\frac{1}{\alpha_f} > (1-\cos\theta) \tag{7}$$

If the denominator tends toward zero, α_i tends toward infinity, so this is equivalent to requiring that the incident photon have finite energy – also very sensible. Solving for θ,

$$\cos\theta > 1-\frac{1}{\alpha_f} \tag{8}$$

$$\theta < \cos^{-1}\left(1-\frac{1}{\alpha_f}\right) \tag{9}$$

In the last line, we reverse the inequality because $\cos\theta$ is a decreasing function of θ as θ increases from 0. Given that α_f must be at least two for pair production,

$$\theta < \cos^{-1}\left(1-\frac{1}{2}\right) = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ \tag{10}$$

Daily problem due 20 Sept 2013: If we wish to observe an object which is 0.25 nm in size, what is the minimum-energy photon which can be used?

Solution: The resolution limit using photons will be - well within an order of magnitue anyway - the wavelength of the photons. If we need a resolution of 0.25 nm, we need a photon of this wavelength, or of energy

$$E = hf = \frac{hc}{\lambda} \approx \frac{1240\,\text{eV}\cdot\text{nm}}{0.25\,\text{nm}} \approx 4960\,\text{eV} = 4.96\,\text{keV} \tag{11}$$

By the way, it is handy to know that $hc \approx 1240\,\text{eV}\cdot\text{nm}$.

The problems below are due by the end of the day on 23 Sept 2013.

1. In Compton scattering, what is the kinetic energy of the electron scattered at an angle φ to the incident photon? Your answer should involve only φ, the incident photon frequency (or energy), and fundamental constants.

Solution: One way is simply to use the electron's energy derived in the notes. In principle, that is it: one has the energy in terms of θ, and a way to get θ from φ, so the energy can be determined from a knowledge of α_i and φ alone. This is acceptable, but inelegant. Finding a direct relationship between energy, α_i, and φ would be much nicer. Recall the dimensionless energy parameters used in the notes:

$$\alpha_i = \frac{\text{incident photon energy}}{\text{electron rest energy}} = \frac{hf_i}{mc^2} \tag{12}$$

$$\alpha_f = \frac{\text{scattered photon energy}}{\text{electron rest energy}} = \frac{hf_f}{mc^2} \tag{13}$$

$$\epsilon = \frac{\text{electron kinetic energy}}{\text{electron rest energy}} = \frac{E_e}{mc^2} \tag{14}$$

The electron's kinetic energy must be the difference between the incident and scattered photon energies:

$$KE_e = hf_i - hf_f = \alpha_i mc^2 - \alpha_f mc^2 = (\alpha_i - \alpha_f)\, mc^2 \tag{15}$$

Solving the Compton equation for α_f, we have

$$\alpha_f = \frac{\alpha_i}{1 + \alpha_i\,(1 - \cos\theta)} \tag{16}$$

Combining these two equations,

$$KE_e = (\alpha_i - \alpha_f)\, mc^2 = mc^2 \left(\alpha_i - \frac{\alpha_i}{1 + \alpha_i\,(1 - \cos\theta)}\right) \tag{17}$$

$$\epsilon = \frac{KE_e}{mc^2} = \frac{\alpha_i^2\,(1 - \cos\theta)}{1 + \alpha_i\,(1 - \cos\theta)} \tag{18}$$

We may use the trigonometric identity $(1 - \cos\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$:

$$\epsilon = \frac{\alpha_i^2\left(2\sin^2\left(\frac{\theta}{2}\right)\right)}{1 + \alpha_i\left(2\sin^2\left(\frac{\theta}{2}\right)\right)} \tag{19}$$

With one more identity, we can put this in terms of $\tan\left(\frac{\theta}{2}\right)$. The next identity is:

$$\sin^2\theta = \frac{\tan^2\theta}{1 + \tan^2\theta} \tag{20}$$

which yields

$$\epsilon = \frac{2\alpha_i^2\left(\dfrac{\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)}{1+2\alpha_i\left(\dfrac{\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)} = \frac{2\alpha_i^2\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)+2\alpha_i\tan^2\left(\frac{\theta}{2}\right)} = \frac{2\alpha_i^2}{\dfrac{1}{\tan^2\left(\frac{\theta}{2}\right)}+1+2\alpha_i} \tag{21}$$

We can make use of another result derived in the notes:

$$\frac{1}{\tan\left(\theta/2\right)} = (1+\alpha_i)\tan\varphi \tag{22}$$

Using this identity, we have the electron energy in terms of φ and α_i alone:

$$\epsilon = \frac{2\alpha_i^2}{1+2\alpha_i+(1+\alpha_i)^2\tan^2\varphi} \tag{23}$$

or

$$E_e = mc^2\left(\frac{2\alpha_i^2}{1+2\alpha_i+(1+\alpha_i)^2\tan^2\varphi}\right) \tag{24}$$

2. Potassium is illuminated with UV light of wavelength 250 nm. **(a)** If the work function of potassium is 2.21 eV, what is the maximum kinetic energy of the emitted electron? **(b)** If the UV light has an intensity of $2\,\mathrm{W/m^2}$, calculate the rate of electron emission per unit area.

Solution: **(a)** The maximum kinetic energy is given by

$$KE_{\max} = hf-\varphi = \frac{hc}{\lambda}-\varphi = \frac{1240\,\mathrm{eV\cdot nm}}{250\,\mathrm{nm}}-2.21\,\mathrm{eV} \approx 2.75\,\mathrm{eV} \tag{25}$$

(b) Our model of the photoelectric effect is one photon in, one electron out. Therefore, if we can figure out how many photons are incident per unit area per unit time, we are done. The beam power is energy per unit time per unit area. If we divide this by the energy per photon, we should have the number of photons per unit time per unit area. The photon energy is hc/λ, so:

$$\begin{aligned}
\mathrm{photons/m^2\cdot s} &= \frac{\text{total light energy}}{\text{time}\cdot\text{area}}\cdot\frac{1}{\text{photon energy}} = \frac{\text{power/area}}{\text{photon energy}} && (26)\\
&= \frac{2\,\mathrm{W/m^2}}{(6.626\times10^{-34}\,\mathrm{J\cdot s})\,(3.00\times10^{8}\,\mathrm{m/s})\,/\,(250\times10^{-9}\,\mathrm{m})} && (27)\\
&\approx 2.52\times10^{18} && (28)
\end{aligned}$$

3. The resolving power of a microscope is proportional to the wavelength used. We desire a 10^{-11} m (0.01 nm) resolution in order to "see" an atom. If electrons are used, what minimum kinetic energy is required to reach this resolution? Do not assume that the electron can be treated without relativity.

Solution: The de Broglie relationship tells us $\lambda = h/p$. In order to resolve features of a certain size with a microscope, the probe we're using should have a wavelength at least the same size as the desired resolution (if not smaller, ideally). In this case, our probe is an electron beam, so we need to have an electron wavelength of at least 10^{-11} m to resolve features of that size. Using the de Broglie relationship, and assuming we may need to consider relativistic effects, we could write

$$\lambda = 10^{-11}\,\mathrm{m} = \frac{h}{p} = \frac{h}{\gamma m v} \tag{29}$$

Of course, we want the kinetic energy, rather than the momentum, so we should make use of the relativistic energy-momentum relationship:

$$K = \sqrt{p^2c^2 + m^2c^4} - mc^2 = (\gamma - 1)\,mc^2 \tag{30}$$

We now have two choices: solve for the speed using Eq. 29, and then calculate K, or solve the whole thing algebraically first to put K in terms of p. We choose the latter.

$$K + mc^2 = \sqrt{p^2c^2 + m^2c^4} \tag{31}$$

$$\left(K + mc^2\right)^2 - m^2c^4 = p^2c^2 \tag{32}$$

$$p = \frac{1}{c}\sqrt{(K + mc^2)^2 - m^2c^4} \tag{33}$$

$$p = \frac{1}{c}\sqrt{\left[(K + mc^2 + mc^2)\,(K + mc^2 - mc^2)\right]} = \frac{1}{c}\sqrt{K\,(K + 2mc^2)} \tag{34}$$

Note the factorization on the last line. Inserting that into Eq. 29, and solving for K:

$$\lambda = \frac{hc}{\sqrt{K\,(K + 2mc^2)}} \tag{35}$$

$$\left(\frac{hc}{\lambda}\right)^2 = K\left(K + 2mc^2\right) \tag{36}$$

$$0 = K^2 + \left(2mc^2\right)K - \left(\frac{hc}{\lambda}\right)^2 \tag{37}$$

$$K = -mc^2 \pm \sqrt{(mc^2)^2 + \left(\frac{hc}{\lambda}\right)^2} \tag{38}$$

Clearly, only the positive root is physical. Note that the positive root always gives a positive kinetic energy, so long as λ is not zero. Using the numbers given, and noting $mc^2 = 511\,\mathrm{keV}$ and $hc = 1240\,\mathrm{eV{\cdot}nm}$,

$$K = \sqrt{(mc^2)^2 + \left(\frac{hc}{\lambda}\right)^2} - mc^2 = \sqrt{(511\,\mathrm{keV})^2 + \left(\frac{1240\,\mathrm{eV}\cdot\mathrm{nm}}{0.01\,\mathrm{nm}}\right)^2} - 511\,\mathrm{keV} \approx 14.8\,\mathrm{keV} \quad (39)$$

This means one needs to accelerate the electron through a potential difference of about 15 kV. Note also

$$K = mc^2\left(\sqrt{1 + \left(\frac{hf}{mc^2}\right)^2} - 1\right) = mc^2\left(\sqrt{1 + \left(\frac{\lambda_c}{\lambda}\right)^2} - 1\right) \quad (40)$$

Here it is more apparent that the relative energy scale is the photon energy hf divided by the electron's rest energy mc^2, and that the relevant distance scale is the electron's wavelength relative to its Compton wavelength $\lambda_c = h/mc$. When the electron's wavelength is of the same order as or smaller than the Compton wavelength, or its energy is comparable to or larger than its rest energy, relativistic and quantum effects become important.

4. By doing a nuclear diffraction experiment, you measure the de Broglie wavelength of a proton to be 9.16 fm. **(a)** What is the speed of the proton? **(b)** Through what potential difference must it be accelerated to achieve that speed?

Solution: Let us assume relativity is needed, since a femtometer is a very small distance scale, implying a large energy. We start with deBroglie, and add relativity:

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m v} = \frac{h\sqrt{1 - v^2/c^2}}{mv} \quad (41)$$

Now we solve for v

$$\lambda^2 m^2 v^2 = h^2 - \frac{h^2 v^2}{c^2} \quad (42)$$

$$h^2 = v^2\left(\lambda^2 m^2 + \frac{h^2}{c^2}\right) \quad (43)$$

$$v = \frac{hc}{\sqrt{h^2 + \lambda^2 m^2 c^2}} = \frac{c}{\sqrt{1 + (\lambda mc/h)^2}} = \frac{c}{\sqrt{1 + (\lambda/\lambda_c)^2}} \approx 0.143c \quad (44)$$

Again we see the relevant distance scale is the Compton wavelength for the proton, $\lambda_c = h/m_p c \approx 1.32 \times 10^{15}\,\mathrm{m}$. Given the speed, the kinetic energy is no big deal, particularly noting that $m_p c^2 \approx$

938 MeV:

$$K = (\gamma - 1)\, mc^2 \approx 9.72\,\text{MeV} \tag{45}$$

Given the proton's charge of $+e$, this means we need to move it through a potential difference of $-9.72\,\text{MV}$ to reach the desired de Broglie wavelength.

5. The Compton shift in wavelength $\Delta\lambda$ is independent of the incident photon energy $E_i = hf_i$. However, the Compton shift in *energy*, $\Delta E = E_f - E_i$ is strongly dependent on E_i. Find the expression for ΔE. Compute the fractional shift in energy for a 10 keV photon and a 10 MeV photon, assuming a scattering angle of 90°.

Solution: The energy shift is easily found from the Compton formula with the substitution $\lambda = hc/E$:

$$\lambda_f - \lambda_i = \frac{hc}{E_f} - \frac{hc}{E_i} = \frac{h}{mc}\,(1 - \cos\theta) \tag{46}$$

$$\frac{cE_i - cE_f}{E_iE_f} = \frac{1 - \cos\theta}{mc} \tag{47}$$

$$\Delta E = E_i - E_f = \left(\frac{E_iE_f}{mc^2}\right)(1 - \cos\theta) \tag{48}$$

$$\frac{\Delta E}{E_i} = \left(\frac{E_f}{mc^2}\right)(1 - \cos\theta) \tag{49}$$

Thus, the fractional energy shift is governed by the photon energy relative to the electron's rest mass, as we might expect. In principle, this is enough: one can plug in the numbers given for E_i and θ, solve for E_f, and then calculate $\Delta E/E_i$ as requested. This is, however, inelegant. One should really solve for the fractional energy change symbolically, being both more elegant and enlightening in the end. Start from Eq. 49 isolate E_f:

$$\frac{E_i - E_f}{E_i} = 1 - \frac{E_f}{E_i} = \frac{E_f}{mc^2}\,(1 - \cos\theta) \tag{50}$$

$$1 = E_f\left[\frac{1}{E_i} + \frac{1}{mc^2}\,(1 - \cos\theta)\right] \tag{51}$$

$$E_f = \frac{1}{1/E_i + (1 - \cos\theta)\,/mc^2} = \frac{mc^2E_i}{mc^2 + E_i\,(1 - \cos\theta)} \tag{52}$$

Now plug that back into the expression for ΔE we arrived at earlier, Eq. 49:

$$\frac{\Delta E}{E_i} = \left(\frac{1}{mc^2}\right)\left(\frac{mc^2 E_i}{mc^2 + E_i\,(1-\cos\theta)}\right)(1-\cos\theta) \tag{53}$$

$$\frac{\Delta E}{E_i} = \frac{E_i\,(1-\cos\theta)}{mc^2 + E_i\,(1-\cos\theta)} = \frac{\left(\frac{E_i}{mc^2}\right)(1-\cos\theta)}{1+\left(\frac{E_i}{mc^2}\right)(1-\cos\theta)} \tag{54}$$

This is even more clear (hopefully): Compton scattering is strongly energy-dependent, and the relevant energy scale is set by the ratio of the incident photon energy to the rest energy of the electron, E_i/mc^2. If this ratio is large, the fractional shift in energy is large, and if this ratio is small, the fractional shift in energy becomes negligible. Only when the incident photon energy is an appreciable fraction of the electron's rest energy is Compton scattering significant. The numerical values required can be found most easily by noting that the electron's rest energy is $mc^2 = 511\,\mathrm{keV}$, which means we don't need to convert the photon energy to joules. One should find:

$$\frac{\Delta E}{E_i} \approx 0.02 \qquad 10\,\mathrm{keV}\text{ incident photon, } \theta = 90^\circ \tag{55}$$

$$\frac{\Delta E}{E_i} \approx 0.95 \qquad 10\,\mathrm{MeV}\text{ incident photon, } \theta = 90^\circ \tag{56}$$

Consistent with our symbolic solution, for the 10 keV photon the energy shift is negligible, while for the 10 MeV photon it is extremely large. Conversely, this means that the electron acquires a much more significant kinetic energy after scattering from a 10 MeV photon compared to a 10 keV photon.

6. A hydrogen atom is moving at a speed of 125.0 m/s. It absorbs a photon of wavelength 97 nm that is moving in the opposite direction. By how much does the speed of the atom change as a result of absorbing the photon?

Solution: Just conservation of momentum. Initially, we have the hydrogen's and photon's momentum, after we have just the hydrogen. We don't need relativity, given the low velocity compared to c.

$$p_{hi} + p_{ph} = p_{hf} \tag{57}$$

Given $p_{ph} = h/\lambda$ and $p_h = mv$, and noting the directions,

$$mv_i - \frac{h}{\lambda} = mv_f \tag{58}$$

$$v_f = v_i - \frac{h}{\lambda m} \tag{59}$$

$$\Delta v = v_f - v_i = \frac{h}{\lambda m} \tag{60}$$

Given a the mass of a hydrogen atom is about 1.67×10^{-27} m, $\Delta v \approx 4.1$ m/s.

7. Suppose an atom of iron at rest emits an X-ray photon of energy 6.4 keV. Calculate the "recoil" momentum and kinetic energy of the atom *Hint: do you expect to need classical or relativistic kinetic energy for the atom? Is the kinetic energy likely to be much smaller than the atom's rest energy?*

Solution: Same as the last problem. Do we need relativity? Only if the photon energy is comparable to the iron atom's rest energy. The latter has a mass of about 9.27×10^{-26} kg, implying a rest energy of $mc^2 \approx 52$ GeV. That's about ten billion times the photon energy, so we are probably good using classical physics. Now it is just conservation of momentum like the last problem: we start with an iron atom at rest ($p = 0$), end up with an iron atom and photon going in the opposite direction. The iron atom and photon must therefore have equal and opposite momentum.

$$p_{Fe} = p_{ph} \tag{61}$$

$$mv = \frac{h}{\lambda} = \frac{1}{c}\frac{hc}{\lambda} = \frac{1}{c}E_{ph} \approx 3.4 \times 10^{-24}\,\mathrm{kg \cdot m/s} \tag{62}$$

This implies a recoil velocity of about 37 m/s. (Note that the thermal velocity at room temperature for an iron atom more like 370 m/s.) The recoil kinetic energy of the iron atom is then

$$K = \frac{p^2}{2m} \approx 3.9 \times 10^{-4}\,\mathrm{eV} = 0.39\,\mathrm{meV} \tag{63}$$

8. *Time delay in the photoelectric effect.* A beam of ultraviolet light of intensity 1.6×10^{-12} W is suddenly turned on and falls on a metal surface, ejecting electrons through the photoelectric effect. The beam has a cross-sectional area of 1 cm^2, and the wavelength corresponds to a photon energy of 10 eV. The work function of the metal is 5 eV. How soon might one expect photoelectric emission to occur? *Note:* $1\,\mathrm{eV} = 1.6 \times 10^{-19}$ J.

(a) One classical model suggests an estimate based on the time needed for the work function energy (5 eV) to be accumulated over the area of one atom (radius ~ 0.1 nm). Calculate how long this would be, assuming the energy of the light beam to be uniformly distributed over its cross section.

(b) Actually, as Lord Rayleigh showed in 1916, the estimate from (a) is too pessimistic. An atom can present an effective area of about λ^2 to light of wavelength λ corresponding to its resonance frequency. Calculate a time delay on this basis.

(c) On the quantum picture of the process, it is possible for photoelectron emission to begin immediately – as soon as the first photon strikes the emitting surface. But to obtain a time that may be compared to the classical estimates, calculate the *average* time interval between arrival of successive 10 eV photons. This would also be the average time delay between switching on the source and getting the first photoelectron. *Hint:* think of the power as photons per unit time.

Solution: The power absorbed by the atom is the fraction of the beam's total area that it intercepts times the total power in the beam. If the beam has power P_b and area A_b, and a circular atom of radius r has an area πr^2, the power absorbed by the atom P_a is

$$P_a = P_b \frac{\pi r^2}{A_b} \tag{64}$$

If the beam power is constant, then so is the power absorbed by the atom. Constant power means constant energy per unit time, so the amount of energy ΔE absorbed in a time Δt by the atom is $\Delta E = P_a \Delta t$, or

$$\Delta t = \frac{\Delta E}{P_a} = \frac{\Delta E A_b}{P_b \pi r^2} \tag{65}$$

The atom needs to absorb an energy of $\Delta E = 5\,\text{eV}$, which will require $\Delta t \approx 1.6 \times 10^9\,\text{s} \sim 50\,\text{yr}$ using the information given. I have it on good authority that this experiment is easily completed in the PH255 laboratory in a few minutes, so something has gone horribly wrong.

Lord Rayleigh used a more accurate cross-section (recall our discussion of cross sections when we analyzed radiation) of λ^2, which in terms of the light energy E is

$$\lambda^2 = \left(\frac{hc}{E}\right)^2 = A_a \tag{66}$$

This leads to

$$\Delta t = \frac{\Delta E}{P_a} = \frac{\Delta E A_b}{P_b A_a} = \frac{\Delta E A_b E^2}{P_b h^2 c^2} \approx 3200\,\text{s} \sim 1\,\text{hr} \tag{67}$$

Better, but still very much wrong.

In the quantum model, the power in the beam is just the number of photons per second times the energy per photon. If we call the number of photons per unit time $\Delta N/\Delta t$, and the energy per photon E

$$P = \frac{\Delta E}{\Delta t} = E \frac{\Delta N}{\Delta t} = 1.6 \times 10^{-12}\,\text{W} \tag{68}$$

This implies $\frac{\Delta N}{\Delta t} \approx 10^6$ photons/sec, or that *on average*, 10^{-6} seconds passes between photons to account for 10^6 arriving over the course of one second.