

## Problem Set 4 Solutions

**Daily problem for 27 Sept** Consider a particle described by the wave function

$$\psi(x) = \frac{N}{x^2 + a^2}$$

- (a) What is the probability  $P(x) dx$  of finding the particle in the interval  $[x, x + dx]$ ?  
 (b) We require that  $\int_{-\infty}^{\infty} P(x) dx = 1$ . What value of  $N$  is required for this to be true?  
 (c) What is the expected value of the particle's position  $\langle x \rangle$

**Solution:** The probability that the particle is in the interval  $[x, x + dx]$  is just the square of the wave function in 1D:

$$P(x) = |\psi|^2 = \frac{N^2}{(x^2 + a^2)^2} \quad (1)$$

Normalization will find the value of  $N$ :

$$1 = \int_{-\infty}^{\infty} \frac{N^2}{(x^2 + a^2)^2} dx = \frac{N^2}{2a^2} \left( \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + \frac{x}{a^2 + x^2} \right) \Bigg|_{-\infty}^{\infty} \quad (2)$$

$$1 = \frac{N^2}{2a^2} \left( \frac{\pi}{2a} - \frac{-\pi}{2a} + 0 - 0 \right) = \frac{N^2 \pi}{2a^3} \quad (3)$$

$$\implies N = \sqrt{\frac{2a^3}{\pi}} \quad (4)$$

With that out of the way, the expected position is easily calculated. By inspection, we can see that the integrand is an odd function in  $x$ , and will integrate to zero.

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx = \int_{-\infty}^{\infty} \frac{x N^2}{(x^2 + a^2)^2} dx = 0 \quad (5)$$

**Daily problem for 30 Sept** Consider the wave function from the preceding problem. (a) Find  $\langle x^2 \rangle$  and  $\Delta x$ . (b) What is the probability the particle is in the interval  $[-a, a]$ ?

**Solution:** The expected value of  $x^2$  is straightforward enough; we will presume you have a table of integrals or Wolfram Alpha handy. We will need the value of  $N$  from the last problem.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi|^2 dx = \int_{-\infty}^{\infty} \frac{x^2 N^2}{(x^2 + a^2)^2} dx = N^2 \frac{\pi}{2a} = \frac{2a^3}{\pi} \frac{\pi}{2a} = a^2 \quad (6)$$

With that in hand,

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{a^2 - 0} = a \quad (7)$$

We can therefore interpret the wave function from the previous problem as one which localizes a particle within a characteristic distance  $a$  of the origin. Of course, you may have recognized the original function ([http://en.wikipedia.org/wiki/Cauchy\\_distribution](http://en.wikipedia.org/wiki/Cauchy_distribution)), in which case this makes sense.

The probability the particle is in a region  $[-a, a]$  can be found by integrating  $P$  from  $-a$  to  $a$ .

$$P(x \in [-a, a]) = \int_{-a}^a |\psi|^2 dx = \int_{-a}^a \frac{2a^3}{\pi} \frac{1}{(x^2 + a^2)^2} dx = \frac{2a^3}{\pi} \frac{1}{2a^3} \left( \tan^{-1} \left( \frac{x}{a} \right) + \frac{ax}{a^2 + x^2} \right) \Bigg|_{-a}^a \quad (8)$$

$$= \frac{1}{\pi} \left( \tan^{-1}(1) - \tan^{-1}(-1) + \frac{a^2}{a^2 + a^2} - \frac{-a^2}{a^2 + a^2} \right) = \frac{1}{\pi} \left( \frac{\pi}{2} + 1 \right) \quad (9)$$

$$= \frac{1}{2} + \frac{1}{\pi} \approx 0.818 \quad (10)$$

**1.** Consider an electron confined to a 1-dimensional box with infinitely high walls. We know that the allowed energies are discrete. However, in order to observe these discrete levels in an experiment, we should expect that their spacing must be large compared to the electron's thermal energy. Presuming we want to be able to resolve the difference between the first two energy levels, and a thermal energy of  $\frac{1}{2}k_bT$ , estimate the size of the largest "box" that our electron can be confined to at temperatures of **(a)** 295 K (room temperature), and **(b)** 4.2 K (boiling point of liquid helium). **(c)** How cold would it have to be to make the box 1 mm wide?

**Solution:** The energy levels of a particle confined to a 1D box are

$$E_n = \frac{n^2 h^2}{8mL^2} \quad n \in \{1, 2, 3, \dots\} \quad (11)$$

The energy difference between the first two levels ( $n=1$  and  $n=2$ ) is then

$$\Delta E_{12} = E_2 - E_1 = \frac{3h^2}{8mL^2} \quad (12)$$

In order to resolve this difference in our experiment, we want the thermal energy to be smaller than

this level spacing. The thermal energy of  $\frac{1}{2}k_bT$  is random, meaning it is essentially an uncertainty on top of any measurement we make. If this thermal energy is larger than the level spacing, it means the uncertainty in our measurement is larger than the difference we're looking for, and we'll have no luck. At a given  $T$ , the thermal uncertainty is fixed, so all we can do is make the box smaller to make the levels easier to resolve. Thus, what we want is

$$\Delta E_{12} \geq \frac{1}{2}k_bT \quad (13)$$

$$\frac{3h^2}{8mL^2} \geq \frac{1}{2}k_bT \quad (14)$$

$$L \leq \sqrt{\frac{3h^2}{2mk_bT}} \quad (15)$$

At  $T = 295$  K, we get a box size of approximately  $L \lesssim 13$  nm, and at 4.2 K we find  $L \lesssim 110$  nm. Both are experimentally achievable these days, though the latter is much, much easier (even indexed against the lower temperature requirement, 4.2 K is not so difficult).

On the other hand, if we want to have a 1 mm box, how cold does our system have to be? Just solve the equation above for  $T$ , with  $L = 10^{-3}$  m.

$$T < \frac{3h^2}{2mL^2k_b} \sim 52 \text{ nK} \quad (16)$$

In fact, this isn't a totally impossible experiment (optical techniques allow cooling to nK temperatures), but clearly it is the case that making anything *macroscopic* exhibit quantum behavior is going to be tricky at best, and not something you'll see every day.

**2.** A particle of mass  $m$  is in the state

$$\psi(x, t) = Ae^{-a[(mx^2/\hbar)+it]} \quad (17)$$

where  $\{A, a\} \in \mathbb{R}$  and  $\{A, a\} > 0$ . **(a)** Find  $A$ . **(b)** For what potential energy function  $V(x)$  does  $\psi$  satisfy the Schrödinger equation? **(c)** Calculate the expected values of  $x$ ,  $x^2$ ,  $p$ , and  $p^2$ . **(d)** Find  $\Delta x$  and  $\Delta p$ . Is their product consistent with the uncertainty principle?

**Solution:** In order to find  $A$ , we have to normalize. First, since we have a complex wavefunction, let's make sure we get the square right. We will assume  $A$  is a positive real constant for convenience.<sup>1</sup>

$$|\psi|^2 = \psi^*\psi = \left(Ae^{-a[(mx^2/\hbar)-it]}\right) \left(Ae^{-a[(mx^2/\hbar)+it]}\right) = A^2e^{-2amx^2/\hbar} \quad (18)$$

---

<sup>1</sup>If  $A$  were complex, we could always absorb the imaginary part into the exponential, which would make it an offset in the  $t$  term. That amounts to a choice of when to zero your clock, which we can always do, so we lose nothing by making the assumption that  $A$  is real.

You didn't forget to do the complex conjugate, right? Now we can normalize, remembering that

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = \int_{-\infty}^{\infty} A^2 e^{-2amx^2/\hbar} dx = A^2 \sqrt{\frac{\pi\hbar}{2am}} \quad (19)$$

$$\implies A = \sqrt[4]{\frac{2am}{\pi\hbar}} \quad (20)$$

What potential energy function gives this wave function? The function in question is a gaussian, and if you read the book carefully you should have seen it already. Let's not spoil the fun though. If we know the wave function, and plug it into the time-dependent Schrödinger's equation, the only unknown should be the potential, so we should be able to solve for it. First, we will need various derivatives of  $\psi$ .<sup>ii</sup> Conveniently, for each of the derivatives you can factor out  $\psi$  to simplify things.

$$\frac{\partial\psi}{\partial x} = -\frac{2am}{\hbar} x\psi \quad (21)$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{2am}{\hbar} \psi + \frac{4a^2m^2}{\hbar^2} x^2\psi = -\frac{2am}{\hbar} \left(1 - \frac{2am}{\hbar} x^2\right) \psi \quad (22)$$

$$\frac{\partial\psi}{\partial t} = -ia\psi \quad (23)$$

Substituting into the time-dependent Schrödinger equation, and solving for the  $V\psi$  term:

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2\psi}{\partial x^2} + V\psi \quad (24)$$

$$V\psi = i\hbar \frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2\psi}{\partial x^2} \quad (25)$$

$$V\psi = a\hbar\psi - \frac{\hbar^2}{2m} \frac{2am}{\hbar} \left(1 - \frac{2am}{\hbar} x^2\right) \psi \quad (26)$$

Now every term still has a  $\psi$  left, so we can cancel all of those.

$$V = a\hbar - a\hbar + 2a^2mx^2 \implies V = 2ma^2x^2 \quad (27)$$

The potential that produces this wavefunction is a quadratic one, i.e., the simple harmonic oscillator potential. Knowing this, we can make the usual identification that  $V = \frac{1}{2}kx^2$ , from which we would deduce  $k = 4ma^2$  and  $\omega = \sqrt{k/m} = 2a$  for the given state of the oscillator.

How about the expectation values? You can quickly convince yourself that  $\langle x \rangle = 0$ , since  $x|\psi|^2$  is an odd function of  $x$ . You could also just assert this to be true - since the potential is symmetric,

---

<sup>ii</sup>Remember, if you are not familiar with partial derivatives, just read " $\partial\psi/\partial t$ " as " $d\psi/dt$ ".

the expected position should be in the center, at  $x=0$ .

$$\langle x \rangle = \int_{-\infty}^{\infty} x |\psi|^2 dx = \int_{-\infty}^{\infty} x A^2 e^{-2amx^2/\hbar} dx = 0 \quad (28)$$

The same will hold for  $\langle p \rangle$ , though again you could assert that the particle must spend as much time going one way as the other in a symmetric potential.

$$\langle p \rangle = \int_{-\infty}^{\infty} \psi^* \frac{\hbar}{i} \frac{\partial \psi}{\partial x} dx = \int_{-\infty}^{\infty} A e^{-a[(mx^2/\hbar)-it]} \frac{\hbar}{i} \left( -\frac{2am}{\hbar} x A e^{-a[(mx^2/\hbar)+it]} \right) dx \quad (29)$$

$$= 2iamA^2 \int_{-\infty}^{\infty} x e^{-2amx^2/\hbar} dx = 0 \quad (30)$$

There is no really clever way to do the other two by inspection, just grind it out.

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 |\psi|^2 dx = \int_{-\infty}^{\infty} x^2 A^2 e^{-2amx^2/\hbar} dx = A^2 \frac{\sqrt{\pi}}{2} \left( \frac{\hbar}{2am} \right)^{3/2} = \frac{\hbar}{4am} \quad (31)$$

The momentum operator is  $(\hbar/i)(d/dx)$ , you can either square that or use  $2m$  times kinetic energy. Either way, it is  $-\hbar^2(d^2/dx^2)$ .

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi^* \left( -\hbar^2 \right) \frac{\partial^2 \psi}{\partial x^2} dx = A^2 \hbar^2 \int_{-\infty}^{\infty} |\psi|^2 \frac{2am}{\hbar} \left( 1 - \frac{2am}{\hbar} x^2 \right) dx \quad (32)$$

$$= 2am\hbar \int_{-\infty}^{\infty} A^2 |\psi|^2 dx - 4a^2 m^2 \int_{-\infty}^{\infty} A^2 x^2 |\psi|^2 dx \quad \text{note 1st term is norm., 2nd is } \langle x^2 \rangle \quad (33)$$

$$= 2am\hbar - 4a^2 m^2 \frac{\hbar}{4am} = am\hbar \quad (34)$$

The uncertainties are then

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{4am} - 0} = \sqrt{\frac{\hbar}{4am}} \quad (35)$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{am\hbar - 0} = \sqrt{am\hbar} \quad (36)$$

$$\Delta x \Delta p = \frac{\hbar}{2} \quad (37)$$

In fact, gaussian wave functions satisfy the *minimum* uncertainty principle allowed.

**3.** Suppose you add a constant  $V_o$  to the potential energy (by “constant” we mean independent of both  $x$  and  $t$ ). In classical mechanics, this doesn’t change anything, but what about quantum me-

chanics? (a) Show that the wave function picks up a time-dependent phase factor:  $\exp(-iV_o t/\hbar)$ . (b) What effect does this have on the expectation value of a dynamical variable like  $x$  or  $p$ ?

**Solution:** Suppose originally  $\psi$  solves the Schrödinger equation without  $V_o$ :

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \quad (38)$$

We want to find  $\psi_o$  such that

$$i\hbar \frac{\partial \psi_o}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_o}{\partial x^2} + (V + V_o)\psi \quad (39)$$

Our claim is that

$$\psi_o = \psi e^{-iV_o t/\hbar} \quad (40)$$

Taking the derivative of the above,

$$i\hbar \frac{\partial \psi_o}{\partial t} = i\hbar \frac{\partial \psi}{\partial t} e^{iV_o t/\hbar} + i\hbar \left( \frac{-iV_o}{\hbar} \right) \psi e^{-iV_o t/\hbar} \quad (41)$$

We know what  $i\hbar \frac{\partial \psi}{\partial t}$  is from Eq. 38

$$i\hbar \frac{\partial \psi_o}{\partial t} = \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi \right) e^{-iV_o t/\hbar} + i\hbar \left( \frac{-iV_o}{\hbar} \right) \psi e^{-iV_o t/\hbar} \quad (42)$$

Since  $\psi_o$  and  $\psi$  have the same dependence on  $x$ , their spatial derivatives only differ by the same factor  $e^{iV_o t/\hbar}$  that  $\psi_o$  and  $\psi$  do:

$$\frac{\partial^2 \psi_o}{\partial x^2} = \frac{\partial^2 \psi}{\partial x^2} e^{-iV_o t/\hbar} \quad (43)$$

Substituting this,

$$i\hbar \frac{\partial \psi_o}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_o}{\partial x^2} + V\psi e^{-iV_o t/\hbar} + V_o\psi e^{-iV_o t/\hbar} \quad (44)$$

But since we claim  $\psi_o = \psi e^{-iV_o t/\hbar}$ ,

$$i\hbar \frac{\partial \psi_o}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi_o}{\partial x^2} + (V + V_o)\psi_o \quad (45)$$

Thus, shifting the overall potential energy by a constant amount  $V_o$  just adds a phase factor  $e^{-iV_o t/\hbar}$

to the wavefunction. This has no effect on the expectation values dynamical variables like  $x$  and  $p$  - since neither the  $x$  nor  $p$  operator involves the time variable, the phase factor will always cancel out when we take complex conjugates and calculate the expectation values. The overall phase of the wave function will cancel out whenever you calculate a measurable quantity.

4. At time  $t=0$  a particle is represented by the wave function

$$\psi(x, 0) = \begin{cases} Ax/a & 0 \leq x \leq a \\ a(b-x)/(b-a) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \quad (46)$$

where  $A$ ,  $a$ , and  $b$  are constants. **(a)** Normalize  $\psi$  (that is, find  $A$  in terms of  $a$  and  $b$ ). **(b)** Sketch  $\psi(x, 0)$  as a function of  $x$ . **(c)** Where is the particle most likely to be found at  $t=0$ ? **(d)** What is the probability of finding the particle to the left of  $a$ ? You can check your result in the limiting cases  $b=a$  and  $b=2a$ . **(e)** What is the expectation value of  $x$ ?

**Solution:** Normalizing means integrating  $|\psi|^2$  over all space, but since  $\psi$  is zero except over  $[0, b]$ , we need only perform the integral over that interval. We will assume  $A$  may be complex, but will presume  $a$  and  $b$  to be real.

$$1 = \frac{|A|^2}{a^2} \int_0^a x^2 dx + \frac{|A|^2}{(b-a)^2} \int_a^b (b-x)^2 dx \quad (47)$$

$$= |A|^2 \left[ \frac{1}{a^2} \left( \frac{x^3}{3} \right) \Big|_0^a + \frac{1}{(b-a)^2} \left( -\frac{(b-x)^3}{3} \right) \Big|_a^b \right] \quad (48)$$

$$= |A|^2 \left[ \frac{a}{3} + \frac{b-a}{3} \right] = |A|^2 \frac{b}{3} \quad (49)$$

$$\implies A = \sqrt{\frac{3}{b}} \quad (50)$$

In the end,  $A$  is purely real, but best to be on the safe side. The wave function is linearly increasing from 0 to  $a$ , going from  $\psi(0) = 0$  to  $\psi(a) = A$ , and linearly decreasing from  $a$  to  $b$ , going from  $\psi(a) = A$  to  $\psi(b) = 0$ . A sketch is left as an exercise to the reader.

The particle is most likely to be found where  $|\psi|^2$  has a maximum, which in this case means the place where  $\psi$  itself is maximum. From the given form (or your sketch), the maximum is clearly at  $x=a$ .

The probability of finding the particle to the left of  $a$  is given by integrating the probability density from  $-\infty$  to  $a$ :

$$P(\text{in } [-\infty, a]) = \frac{|A|^2}{a^2} \int_0^a x^2 dx = \frac{3}{ba^2} \left. \frac{x^3}{3} \right|_0^a = \frac{a}{b} \quad (51)$$

In the limiting case that  $b=a$ ,  $P=1$ . In this case the wavefunction is zero for  $x>a$ , so the particle has nowhere else to be. In the limiting case  $b=2a$ , the wavefunction is defined over two symmetric intervals, with  $x=a$  right in the middle. By symmetry, the particle must be to the left of  $a$  half of the time, and we indeed find  $P=\frac{1}{2}$ .

The expectation value is found by integrating  $x|\psi|^2$  over all space, or over  $[0, b]$  since  $\psi$  is zero elsewhere.

$$\langle x \rangle = \int x|\psi|^2 dx = \frac{|A|^2}{a^2} \int_0^a x^3 dx + \frac{|A|^2}{(b-a)^2} \int_a^b x(b-x)^2 dx \quad (52)$$

$$= \frac{3}{b} \left[ \frac{1}{a^2} \left( \frac{x^4}{4} \right) \Big|_0^a + \frac{1}{(b-a)^2} \left( b^2 \frac{x^2}{2} - 2b \frac{x^3}{3} + \frac{x^4}{4} \right) \Big|_a^b \right] \quad (53)$$

$$= \frac{3}{4b(b-a)^2} \left[ a^2(b-a)^2 + 2b^4 - \frac{8b^4}{3} - 2a^2b^2 + \frac{8a^3b}{3} - a^4 \right] \quad (54)$$

$$= \frac{3}{4b(b-a)^2} \left( \frac{b^3}{3} - a^2b^2 + \frac{2}{3}a^3b \right) = \frac{1}{4(b-a)^2} (b^3 - 3a^2b + 2a^3) = \frac{2a+b}{4} \quad (55)$$

Wolfram Alpha will tell you that  $b^3 - 3a^2b + 2a^3$  factors to  $(a-b)^2(2a+b)$  if you didn't notice. The last step is only valid if  $a \neq b$ , which seems sensible based on the definition of the wavefunction given.

**5. (a)** For surface tension waves in shallow water, the relation between frequency and wavelength is given by

$$f = \sqrt{\frac{2\pi T}{\rho\lambda^3}} \quad (56)$$

where  $T$  is the surface tension and  $\rho$  the density. What is the group velocity of the waves, and its relation to the phase velocity, defined to be  $v_p = \lambda f$ ? **(b)** For gravity waves (deep water), the relation is given by

$$f = \sqrt{\frac{g}{2\pi\lambda}} \quad (57)$$

What are the group and phase velocities?

**Solution:** This will be easier if we write the relationships in terms of  $\omega = 2\pi f$  and  $k = 2\pi/\lambda$ , since



we know  $v_g = d\omega/dk$  and  $v_p = \omega/k$ . In the first case,

$$\frac{\omega}{2\pi} = \sqrt{\frac{2\pi T}{\rho (2\pi/k)^3}} = \sqrt{\frac{Tk^3}{\rho (2\pi)^2}} \quad (58)$$

$$\implies \omega = \sqrt{\frac{Tk^3}{\rho}} \quad (59)$$

This leads us to

$$v_g = \frac{d\omega}{dk} = \frac{3}{2} \sqrt{\frac{kT}{\rho}} \quad (60)$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{Tk}{\rho}} \quad (61)$$

Both velocities are in the same direction, but the group velocity is a factor of 3/2 faster than the phase velocity. For the deep water case, we find

$$\frac{\omega}{2\pi} = \sqrt{\frac{g}{2\pi (2\pi/k)}} = \frac{1}{2\pi} \sqrt{gk} \quad (62)$$

$$\implies \omega = \sqrt{gk} \quad (63)$$

This leads us to

$$v_g = \frac{d\omega}{dk} = \frac{1}{2} \sqrt{\frac{g}{k}} \quad (64)$$

$$v_p = \frac{\omega}{k} = \sqrt{\frac{g}{k}} \quad (65)$$

Again both velocities are in the same direction, but now the *phase* velocity is the larger of the two.

A very interesting discussion of the consequences of the differing behavior of surface and deep water waves can be found in the Feynman Lectures (I.51), [http://www.feynmanlectures.caltech.edu/I\\_51.html#Ch51-S4](http://www.feynmanlectures.caltech.edu/I_51.html#Ch51-S4).

**6.** Nuclei, typically of size  $10^{-14}$  m, frequently emit electrons, with typical energies of 1–10 MeV. Use the uncertainty principle to show that electrons of energy 1 MeV could not be contained in the nucleus before the decay.

**Solution:** Clearly we need to use the uncertainty principle. We are given a characteristic region in which the electron is confined, which we can interpret as the minimum uncertainty in position,  $\Delta x = 10^{-14}$  m. The uncertainty principle will get us the corresponding  $\Delta p$ , which we can relate to the electron's total energy. We're obliged to use the relativistic energy-momentum relationship,

since the energy in question is far above the electron's rest energy ( $m_e c^2 \approx 511 \text{ keV}$ ).

One more question: should we use total energy or kinetic energy? For the purposes of this problem and our general conclusion, it doesn't make much difference, but let us assume that what experimenters are able to measure is the kinetic energy. This amounts to saying that the rest energy is unchanged before and after decay, so it would cancel.

First, we can find the minimum uncertainty in momentum:

$$\Delta x \Delta p \geq \frac{\hbar}{2} \quad \implies \quad \Delta p \geq \frac{\hbar}{2\Delta x} \quad (66)$$

If  $\Delta p$  is the minimum *uncertainty* in momentum, then this is also the minimum value for momentum:  $p_{\min} = \Delta p$ . Noting that, and using the relativistic kinetic energy-momentum relationship (total energy minus rest energy),

$$KE_{\min} = \sqrt{m^2 c^4 + p_{\min}^2 c^2} - mc^2 = \sqrt{m^2 c^4 + \frac{\hbar^2 c^2}{4(\Delta x)^2}} - mc^2 \approx 62 \text{ MeV} \quad (67)$$

An electron would require a kinetic energy of at least 62 MeV to be confined to a region of  $10^{-14} \text{ m}$ , far more than the 1–10 MeV the observed electrons have. This is a big hint as to how nuclear decay has to work ...

Incidentally, knowing that the electron's energy is well above its rest energy, we could neglect the  $m^2 c^4$  and  $mc^2$  rest energy terms, and simplify things a bit. This amounts to saying  $K \approx pc$  for highly relativistic electrons.

$$KE_{\min} \approx p_{\min}^2 c^2 = \frac{\hbar c}{2\Delta x} \approx 62 \text{ MeV} \quad (68)$$

We can also pose the question another way: what would the confinement size  $\Delta x$  have to be to be consistent with the observed electron energies? We just have to solve for  $\Delta x$  instead.

$$\Delta x = \frac{\hbar c}{\sqrt{4 \left[ (KE_{\min} + mc^2)^2 - m^2 c^4 \right]}} \quad (69)$$

If we know  $KE_{\min} \gg mc^2$ , we can simplify this to  $\Delta x \approx \hbar c / 2KE_{\min}$  to very good accuracy. For 10 MeV electrons, we find  $\Delta x \approx 6 \times 10^{-14} \text{ m}$ , nearly an order of magnitude larger (and we know nuclei aren't this large). For 1 MeV electrons, we find  $\Delta x \approx 4 \times 10^{-13} \text{ m}$ , nearly two orders of magnitude larger.

An interesting read related to this problem, and fundamental length scales in general can be found here: <http://math.ucr.edu/home/baez/lengths.html>.

**7. Zero point energy of a harmonic oscillator.** The frequency  $f$  of a harmonic oscillator of mass  $m$  and elasticity constant  $k$  is given by the equation

$$f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad (70)$$

The energy of the oscillator is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \quad (71)$$

where  $p$  is the system's linear momentum and  $x$  is the displacement from its equilibrium position. Use the uncertainty principle,  $\Delta x \Delta p \approx \hbar/2$ , to express the oscillator's energy  $E$  in terms of  $x$  and show, by taking the derivative of this function and setting  $dE/dx=0$ , that the minimum energy of the oscillator (its ground state energy) is  $E_{\min} = \hbar f/2$ .

**Solution:** The minimum uncertainty in momentum  $\Delta p$ , given an uncertainty  $\Delta x$  in position is given by the uncertainty principle:

$$\Delta x \Delta p = \frac{\hbar}{2} \quad \implies \quad \Delta p = \frac{\hbar}{2\Delta x} \quad (72)$$

The minimum uncertainty is also then the minimum average value we can expect either variable to take on:  $p_{\min} = \Delta p \equiv p$ ,  $x_{\min} = \Delta x \equiv x$ . The energy equation may be rewritten in terms of the minimal  $x$  and  $p$ :

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}kx^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}m\omega^2 x^2 \quad (73)$$

In the last line, we used  $\omega = \sqrt{k/m}$ , so  $k = m\omega^2$ . Minimizing the energy with respect to  $x$ ,

$$\frac{dE}{dx} = \frac{-2\hbar^2}{4mx^3} + m\omega^2 x = 0 \quad \implies \quad x^2 = \frac{\hbar}{2m\omega} \quad (74)$$

Plugging this back in to the energy equation, we have the minimum energy:

$$E_{\min} = \frac{\hbar^2}{8m} \frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2 \frac{\hbar}{2m\omega} = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega \quad (75)$$