Problem Set 4

Instructions:

1. Answer all questions below. Show your work for full credit.
2. The first problem is due at the start of class on 27 Sept 2013
3. The second problem is due at the start of class on 30 Sept 2013
4. The remaining problems are due by the end of the day on 2 Oct 2013
5. You may collaborate, but everyone must turn in their own work.

Daily problem for 27 Sept Consider a particle described by the wave function

$$\psi(x) = \frac{N}{x^2 + a^2}$$

(a) What is the probability $P(x)\,dx$ of finding the particle in the interval $[x, x+dx]$?
(b) We require that $\int_{\infty}^{\infty} P(x)\,dx = 1$. What value of N is required for this to be true?
(c) What is the expected value of the particle's position $\langle x \rangle$

Daily problem for 30 Sept Consider the wave function from the preceding problem. **(a)** Find $\langle x^2 \rangle$ and Δx. **(b)** What is the probability the particle is in the interval $[-a, a]$?

1. Consider an electron confined to a 1-dimensional box with infinitely high walls. We know that the allowed energies are discrete. However, in order to observe these discrete levels in an experiment, we should expect that their spacing must be large compared to the electron's thermal energy. Presuming we want to be able to resolve the difference between the first two energy levels, and a thermal energy of $\frac{1}{2}k_bT$, estimate the size of the largest "box" that our electron can be confined to at temperatures of **(a)** 295 K (room temperature), and **(b)** 4.2 K (boiling point of liquid helium). **(c)** How cold would it have to be to make the box 1 mm wide?

2. A particle of mass m is in the state

$$\psi(x,t) = Ae^{-a\left[(mx^2/\hbar)+it\right]} \tag{1}$$

where $\{A, a\} \in \mathbb{R}$ and $\{A, a\} > 0$. **(a)** Find A. **(b)** For what potential energy function $V(x)$ does ψ satisfy the Schr odinger equation? **(c)** Calculate the expected values of x, x^2, p, and p^2. **(d)** Find Δx and Δp. Is their product consistent with the uncertainty principle?

Make use of symmetry – e.g., are you integrating even or odd functions?

3. Suppose you add a constant V_o to the potential energy (by "constant" we mean independent of both x and t). In classical mechanics, this doesn't change anything, but what about quantum mechanics? **(a)** Show that the wave function picks up a time-dependent phase factor: $\exp(-iV_o t/\hbar)$. **(b)** What effect does this have on the expectation value of a dynamical variable like x or p?

4. At time $t=0$ a particle is represented by the wave function

$$\psi(x,0) = \begin{cases} Ax/a & 0 \leqslant x \leqslant a \\ A(b-x)/(b-a) & a \leqslant x \leqslant b \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where A, a, and b are constants. **(a)** Normalize ψ (that is, find A in terms of a and b). **(b)** Sketch $\psi(x,0)$ as a function of x. **(c)** Where is the particle most likely to be found at $t=0$? **(d)** What is the probability of finding the particle to the left of a? You can check your result in the limiting cases $b=a$ and $b=2a$. **(e)** What is the expectation value of x?

5. **(a)** For surface tension waves in shallow water, the relation between frequency and wavelength is given by

$$f = \sqrt{\frac{2\pi T}{\rho\lambda^3}} \tag{3}$$

where T is the surface tension and ρ the density. What is the group velocity of the waves, and its relation to the phase velocity, defined to be $v_p = \lambda f$? **(b)** For gravity waves (deep water), the relation is given by

$$f = \sqrt{\frac{g}{2\pi\lambda}} \tag{4}$$

What are the group and phase velocities?

6. Nuclei, typically of size 10^{-14} m, frequently emit electrons, with typical energies of $1-10$ MeV. Use the uncertainty principle to show that electrons of energy 1 MeV could not be contained in the nucleus before the decay.

7. *Zero point energy of a harmonic oscillator.* The frequency f of a harmonic oscillator of mass m and elasticity constant k is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{5}$$

The energy of the oscillator is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{6}$$

where p is the system's linear momentum and x is the displacement from its equilibrium position. Use the uncertainty principle, $\Delta x \Delta p \approx \hbar/2$, to express the oscillator's energy E in terms of x and show, by taking the derivative of this function and setting $dE/dx = 0$, that the minimum energy of the oscillator (its ground state energy) is $E_{\text{min}} = hf/2$.