Problem Set 7 Solutions

1. The specific heat at constant volume *per electron* is defined as

$$c_V = \frac{d\langle E\rangle}{dT} \tag{1}$$

where $\langle E\rangle$ is the average energy of electrons. A classical electron would have an average energy of $\frac{3}{2}k_BT$. Presuming the electrons follow a Fermi-Dirac distribution, estimate the specific heat for a collection of free electrons. Presume low temperatures, i.e., $k_BT \ll E_F$. *Hint: electrons that have been thermally excited above the Fermi energy behave basically as free electrons. What fraction of electrons are excited above E_F, roughly?*

Solution: Roughly speaking, only electrons within a few k_BT of the Fermi energy E_F will contribute to the specific heat, because only those electrons will have available states at a nearby energy. How many k_BT should we take? We could characterize the 'width' of the distribution $f(E)$ by taking df/dE (a function peaked at E_F) and finding its full width at half maximum, which is about $3.5k_BT$. About half of that width (the upper half) ought to be electrons available. (Or you could have just said "about three k_BT" and it amounts to the same thing.)The fraction of electrons contributing then would be

$$\text{fraction} = \frac{3.5k_BT}{2E_F} \tag{2}$$

The average energy of these electrons would then be the classical result $\frac{3}{2}k_BT$ times the fraction of electrons actually available we just found:

$$\langle E\rangle = \frac{3}{2}k_BT\left(\frac{3.5k_BT}{2E_F}\right) = \frac{21}{8}\frac{k_B^2T^2}{E_F} \tag{3}$$

The specific heat is then approximately

$$c_V = \frac{d\langle E\rangle}{dT} \approx \frac{21}{4}\frac{k_B^2T}{E_F} = 5.25\frac{k_B^2T}{E_F} \tag{4}$$

We find c_V linear in T, which agrees with experiments at low temperature. The exact result using the full Fermi-Dirac distribution is $\frac{\pi^2}{2}\frac{k_B^2T}{E_F} \approx 4.9\frac{k_B^2T}{E_F}$, off by about 7%. Of course, we could just have easily decided to use 2 or 3 k_BT as our fraction, so the agreement is a bit spurious, but the facts that $c_V \propto T$ and that we have the right order of magnitude still hold true.

2. Following section 10.6 in your textbook (find one), you can find the total energy of a photon gas

U as well as the total number of photons in the gas N.[i] Recall that a photon gas was our model for blackbody radiation! **(a)** What is the *total* specific heat C_V for the photon gas, noting

$$C_V = \frac{\partial U}{\partial T} \tag{5}$$

(b) How does the average energy per photon (U/N) vary with temperature? **(c)** Given this average energy per photon, estimate the specific heat *per photon* using the expression given in problem 1.

Solution: The internal energy of a photon gas of volume V, using the link in the footnotes, is

$$U = \frac{8\pi^5 k_B^4}{15c^3h^3} VT^4 \tag{6}$$

You'd find this by integrating the Planck formula for the energy density over all frequencies, by the way. The total specific heat (or heat capacity) is then

$$C_V = \frac{\partial U}{\partial T} = \frac{32\pi^5 k_B^4}{15c^3h^3} VT^3 \tag{7}$$

The same link gives us the expected number of photons in the gas:

$$N = \frac{16\pi k_B^3 \zeta(3)}{c^3h^3} VT^3 \tag{8}$$

where $\zeta(n)$ is the Riemann zeta function, and $\zeta(3) \approx 1.202$. The average energy per photon then goes as

$$\frac{U}{N} = \frac{\pi^4 k_B T}{30\zeta(3)} \tag{9}$$

The specific heat per photon is then readily found.

$$c_V = \frac{\partial (U/N)}{\partial T} = \frac{\pi k_B}{240\zeta(3)} \tag{10}$$

The heat capacity per photon is a constant, independent of temperature.

3. Use the free electron theory to determine the Fermi energy and density of states vs. energy for a *two-dimensional* metal. Take N as the average number of electrons per unit area.

Solution: In two dimensions, we can write the energy as a function of $\vec{\mathbf{k}}$ in terms of the x and y

[i] For N, use the expression for dN in Eq. 10.38 and integrate it over all energies. Or look here: `http://en.wikipedia.org/wiki/Photon_gas`. Always read the footnotes.

components of the k vector:

$$E(\vec{\mathbf{k}}) = \frac{\hbar^2}{2m}\left(k_x^2 + k_y^2\right) \tag{11}$$

What we need to find is the density of allowed k values. Given a crystal of side L, our boundary conditions dictate that the k components must come in integer multiples of $2\pi/L$. If we made a plot with axes k_x and k_y, there would be allowed states at every integer multiple of $2\pi/L$ along each axis, making a square lattice of such points. The area each allowed state takes up is then $dA = (2\pi/L)^2$. That's how much "k-space" each allowed state takes up.

What about the total number of states and their area? We have as many states as it takes to fill all of them up to the Fermi wave vector k_F, so all the allowed states lie within a circle of radius k_F. Their total area is thus $A = \pi k_F^2$. The number of states is then the total area of all states divided by the area of a single state, times 2 to account for the fact that each state can have one spin up and one spin down electron:

$$N = 2\frac{A}{dA} = 2\frac{\pi k_F^2}{(2\pi/L)^2} = \frac{k_F^2 L^2}{2\pi} \tag{12}$$

Inverting, we can find what k_F must be given N electrons in a square of side L in two dimensions:

$$|\vec{\mathbf{k}}_F| = k_F = \frac{\sqrt{2\pi N}}{L} \tag{13}$$

The Fermi energy is then readily found from the energy-wavevector relationship

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\pi N \hbar^2}{mL^2} \tag{14}$$

How about the density of states? From the preceding equation, at an energy E we know what the energy is as a function of N by virtue of knowing $E(k)$ and $k(N)$. Equation 13 tells us that for a given N, $k^2 = 2\pi N/L^2$, thus

$$E(N) = \frac{\hbar^2}{2m}\left(k(N)\right)^2 = \frac{\hbar^2}{2m}\left(\frac{2\pi N}{L^2}\right) = \frac{\pi \hbar^2 N}{mL^2} \tag{15}$$

The density of states per unit volume just depends on dE/dN and the volume $V = L^2$:

$$g(E) = \frac{1}{V}\frac{dN}{dE} = \frac{1}{L^2}\frac{1}{dE/dN} = \frac{1}{L^2}\frac{mL^2}{\pi\hbar^2} = \frac{m}{\pi\hbar^2} \tag{16}$$

In 2D, the density of states is a constant, independent of energy.

4. **(a)** Obtain an expression for the Fermi energy at $T = 0\,\mathrm{K}$ for an electron gas in a three dimensional metal in terms of the total number of electrons, the volume, and fundamental constants. **(b)** At $T = 0\,\mathrm{K}$, what is the average speed, in terms of the Fermi energy, of a three-dimensional electron gas in a metal?[ii]

Solution: The Fermi energy at $T = 0$ is given by

$$E_F|_{T=0} = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \tag{17}$$

where N is the number of electrons in a volume V. A derivation may be found in your textbook, or here:

`http://en.wikipedia.org/wiki/Fermi_energy`

This is the energy of the highest-energy electrons in a metal at $T = 0$. If all this energy is available as kinetic energy,

$$E_f = \frac{1}{2}mv^2 \qquad \Longrightarrow \qquad v = \sqrt{\frac{2E_F}{m}} \tag{18}$$

5. Show that the average kinetic energy of an electron in a three-dimensional electron gas at $0\,\mathrm{K}$ is $E_{\mathrm{av}} = \frac{3}{5}E_F$.[iii]

Solution: The Fermi energy, the energy of the highest occupied quantum state in a system of fermions at absolute zero temperature, is a function of the number of fermions as found in the last problem. If there are N particles in a volume V, then the highest occupied state has an energy

$$E_F(T = 0) = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \tag{19}$$

Thus, as more and more particles are added, it takes more and more energy to add the last particle. The total energy of the system, if there are enough particles to consider the distribution of states to be quasi-continuous, is given by

$$E_{\mathrm{tot}} = \int_0^N E_F(N')\,dN' \tag{20}$$

[ii] Open your textbook to 10.7.
[iii] Open your textbook to 10.7.

The average energy is simply the total energy divided by the number of particles:

$$E_{\text{avg}} = \frac{1}{N}\int_0^N E_F(N')\,dN' = \frac{1}{N}\int_0^N \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N'}{V}\right)^{2/3} dN' = \frac{1}{N}\frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3}\left(\frac{3}{5}N^{5/3}\right) = \frac{3}{5}E_F \quad (21)$$

6. Now that you've followed the derivation for the three dimensional case from the textbook, repeat the previous **two** problems for a two dimensional electron gas, using your results from problem 3.

Solution: We already found the Fermi energy in problem 3, conveniently enough.

$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\pi N \hbar^2}{mL^2} \quad (22)$$

The average speed at the Fermi energy is the same as in three dimensions (in terms of E_F anyway):

$$v = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2\pi N\hbar^2}{m^2L^2}} = \sqrt{2\pi N}\frac{\hbar}{mL} \quad (23)$$

If the density of states is constant, the average energy is easy - it must be half the maximum value, $\frac{1}{2}E_F$. As a sanity check, we can proceed as in the previous problem:

$$E_{\text{avg}} = \frac{1}{N}\int_0^N E_F(N')\,dN' = \frac{1}{N}\int_0^N \frac{\pi N'\hbar^2}{mL^2}\,dN' = \frac{1}{N}\frac{\pi N^2\hbar^2}{2mL^2} = \frac{\pi N\hbar^2}{2mL^2} = \frac{1}{2}E_F \quad (24)$$