University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair Spring 2010

Problem Set 1: Solutions

Instructions:

1. Answer all questions below.
2. Show your work for full credit.
3. All problems are due Thurs 21 January 2010 by the end of the day.
4. You may collaborate, but everyone must turn in their own work.

1. *Pfeffer & Nir, Prob. 5* The radius of the circular path of an electron moving with a velocity v at right angles to a magnetic field B is given classically by

$$r = \frac{mv}{eB} \tag{1}$$

This equation is valid for $v \ll c$.

(a) What does the relativistic version of this formula look like, valid for all speeds.
(b) Calculate the radius of the path of an electron with an energy of 10 MeV moving at right angles to a magnetic field strength of $B = 2\,\text{T}$.

The main issue with this problem is that the familiar $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$is no longer valid in relativity, and that invalidates our usual approach. Instead, we need to go back to a more general version of force, viz.,

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} \tag{2}$$

In the normal Newtonian regime, where $\vec{\mathbf{p}} = m\vec{\mathbf{v}}$, this does equate to $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$.[i] Relativistically, momentum is defined as $\vec{\mathbf{p}} = \gamma m\vec{\mathbf{v}}$. Force can then be found with a bit of calculus, assuming m is constant:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = \frac{d}{dt}\left(\gamma m\vec{\mathbf{v}}\right) = m\frac{d}{dt}\left(\gamma\vec{\mathbf{v}}\right) = m\gamma\frac{d\vec{\mathbf{v}}}{dt} + m\vec{\mathbf{v}}\frac{d\gamma}{dt} \tag{3}$$

[i] So long as the mass of the object is constant, which it usually is.

This is our equivalent of $\vec{\mathbf{F}}=m\vec{\mathbf{a}}$, and if the magnetic force on the charge q is the only one present, we have

$$\vec{\mathbf{F}} = m\gamma\frac{d\vec{\mathbf{v}}}{dt} + m\vec{\mathbf{v}}\frac{d\gamma}{dt} = q\vec{\mathbf{v}} \times \vec{\mathbf{B}} \tag{4}$$

In order to proceed further, we need to consider the physical situation. First, we know that the magnetic field and velocity are at right angles, so we may simplify the vector product and rewrite this as a friendlier scalar equation:

$$m\gamma\frac{dv}{dt} + mv\frac{d\gamma}{dt} = qvB \tag{5}$$

Additionally, we know that a particle moving in a circular path has a constant *speed*, only the direction of velocity changes, not its magnitude. If speed is constant, then so is γ, since γ depends only on v^2, or $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$, which is just the squared magnitude of velocity or squared speed. If γ is constant, then its time derivative vanishes, allowing some simplification:

$$m\gamma\frac{dv}{dt} = qvB \tag{6}$$

The velocity does change, at least in direction, so its time derivative does not vanish. In principle, we could just solve this. However, the fact that the motion is circular at constant speed provides additional constraint on the motion. Normally, we call this centripetal acceleration, and it is a purely geometrical constraint.[ii] The constraint on acceleration due to the fact that the path is circular does not depend on whether we take a a relativistic point of view or not, but the *force* required to maintain the path *does* because $F = ma$ is no longer valid. In order to maintain a circular path of radius r, the magnitude of the required acceleration is

$$a = \frac{dv}{dt} = \frac{v^2}{r} \tag{7}$$

Again, the subtlety here is that the required *acceleration* is the same as we would have found in introductory mechanics, it is just that our notion of *force* is different in relativity. This constraint allows us to simplify our force equation:

$$m\gamma\frac{dv}{dt} = m\gamma\frac{v^2}{r} = qvB \qquad \Longrightarrow \qquad r = \frac{\gamma mv}{qB} = \frac{p}{qB} \tag{8}$$

Thus, the radius expected from a purely classical analysis is a factor γ too small when we account

[ii] See, for example, `http://faculty.mint.ua.edu/~pleclair/ph125/Notes/curved_paths.pdf` for a derivation and a review of motion on curved paths.

for relativistic effects, but it is still equal to the momentum divided by qB. Next, we need to determine the particle's momentum from its energy in order to obtain a numerical answer from the given quantities, which we can do with the relativistic energy-momentum relationship:

$$E^2 = p^2c^2 + m^2c^4 \qquad \Longrightarrow \qquad p = \frac{1}{c}\sqrt{E^2 - m^2c^4} \tag{9}$$

Putting it all together,

$$r = \frac{\gamma mv}{qB} = \frac{p}{qB} = \frac{\sqrt{E^2 - m^2c^4}}{cqB} \tag{10}$$

Lastly, there is a question of units. The electron's energy is quoted as 10 MeV, which means 10^7 eV or ten million electron volts. An electron volt is the energy one electron acquires when accelerated through a potential difference of 1 V, and thus $1\,\text{eV} = 1.60 \times 10^{-19}$ J. With energy in joules, mass in kilograms, c in meters per second, q in coulombs, and B in tesla, you should find $r \approx 16\,\text{mm}$

2. A classic "paradox" involving length contraction and the relativity of simultaneity is as follows: Suppose a runner moving at $0.75c$ carries a horizontal pole 15 m long toward a barn that is 10 m long. The barn has front and rear doors. An observer on the ground can instantly and simultaneously open and close the two doors by remote control. When the runner and the pole are inside the barn, the ground observer closes and then opens both doors so that the runner and pole are momentarily captured inside the barn and then proceed to exit the barn from the back door. Do both the runner and the ground observer agree that the runner makes it safely through the barn?

See, for example:

http://hyperphysics.phy-astr.gsu.edu/HBASE/Relativ/polebarn.html
http://en.wikipedia.org/wiki/Ladder_paradox

(The links should be clickable.)

3. An astronaut takes a trip to Sirius, which is located a distance of 8 light-years from the Earth. The astronaut measures the time of the one-way journey to be 6 yr. If the spaceship moves at a constant speed of $0.8c$, how can the 8-ly distance be reconciled with the 6-yr trip time measured by the astronaut?

The 8 light-year distance is that measured according to the stationary observers, viz., the earthlings. According to the astronaut, who is in motion relative to Earth and Sirius, the distance is shortened by a factor γ:

$$L_{\text{astronaut}} = \frac{1}{\gamma} L_{\text{earthlings}} = (8\,\text{light-years}) \left(\sqrt{1 - (0.8c)^2/c^2} \right) = (8\,\text{light-years})\,(0.6) = 4.8\,\text{light-years} \tag{11}$$

The astronaut measures the trip to take 6 yr, which means the astronaut would report a velocity of

$$v = \frac{4.8\,\text{light-years}}{6\,\text{yr}} = 0.8c \tag{12}$$

Thus, there is no paradox: the difference in measured times is due to time dilation/length contraction. More to the point: we can't divide one observers distance by another observer's time and expect to get sensible answers unless they are in the same reference frame!

4. *The red shift.* A light source recedes from an observer with a speed $v_{\text{source}} << c$.
(a) Show that the fractional shift in the measured wavelength is given by the approximate expression

$$\frac{\Delta\lambda}{\lambda} \approx \frac{v_{\text{source}}}{c} \tag{13}$$

This phenomenon is known as the red shift, because the visible light is shifted toward the red.
(b) Spectroscopic measurements of light at $\lambda = 397\,\text{nm}$ coming from a galaxy in Ursa Major reveal a red shift of 20.0 nm. What is the recessional speed of the galaxy?

It might be easier to think about this problem in terms of flashes of light rather than individual light waves. Imagine we have a light attached to a spaceship, passing by us at relativistic speed v, emitting flashes of light at regular intervals Δt seconds long. At the instant the spaceship passes by us, we see a flash of light emitted and the ship continues moving away from us at a constant speed. When will we see the next flash? The spaceship's clock is running slow according to us, due to time dilation, so compared to the rate of flashing as measured on the spaceship Δt it will take

$$\Delta t'_1 = \gamma \Delta t = \frac{\Delta t}{\sqrt{1 - v^2/c^2}} \qquad \text{time before next flash emitted} \tag{14}$$

seconds before we see the next flash. However, the ship is moving away from us, so we won't actually see the next flash until the light has traveled back to us over the distance covered by the spaceship between flashes, which is still Δt according to its clock. According to the spaceship, the distance it covers during that time Δt is $d = v\Delta t$, while according to us it must be

$$d' = v\Delta t'_2 = \gamma d = \frac{v\Delta t}{\sqrt{1 - v^2/c^2}} \tag{15}$$

after taking into account length contraction. The light coming back to us covers this distance at a velocity c, so the time required is

$$\Delta t'_2 = \frac{d'}{c} = \frac{\gamma d}{c} = \frac{v\Delta t}{c\sqrt{1 - v^2/c^2}} \qquad \text{time for flash to travel back to us} \tag{16}$$

The total time between our observing the first flash as the spaceship passes by and the second flash being emitted and reaching us is then

$$\Delta t'_{\text{total}} = \Delta t'_1 + \Delta t'_2 = \frac{\Delta t}{\sqrt{1 - v^2/c^2}} + \frac{v\Delta t}{c\sqrt{1 - v^2/c^2}} = \Delta t \left(\frac{1 + v/c}{\sqrt{1 - v^2/c^2}} \right) = \Delta t \sqrt{\frac{1 + v/c}{1 - v/c}} \tag{17}$$

The change in frequency is just the reciprocal of the time interval, so

$$f' = f\sqrt{\frac{1 - v/c}{1 + v/c}} \qquad \text{source receding from observer} \tag{18}$$

This is valid for a source receding from the observer. If the source and observer are moving closer together, then we simply take $v < 0$:

$$f' = f\sqrt{\frac{1 + v/c}{1 - v/c}} \qquad \text{source approaching observer} \tag{19}$$

The wavelength is easily found from the frequency, noting $\lambda f = c$:

$$\lambda' = \frac{c}{f'} = \frac{c}{f}\sqrt{\frac{1 + v/c}{1 - v/c}} = \lambda\sqrt{\frac{1 + v/c}{1 - v/c}} \qquad \text{source receding from observer} \tag{20}$$

The fractional change in wavelength can then be calculated:

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1 = \left(1 + \frac{v}{c}\right)^{1/2} \left(1 - \frac{v}{c}\right)^{-1/2} - 1 \tag{21}$$

$$\approx \left(1 + \frac{v}{2c}\right) \left(1 + \frac{v}{2c}\right) - 1 = \frac{v}{c} - \frac{v^2}{4c^2} \approx \frac{v}{c} \qquad (v \ll c) \tag{22}$$

Thus, the fractional change in wavelength is approximately the source speed relative to c, for $v \ll c$.

Moreover, the wavelength should be *larger* when the source and observer are moving away from each other. This is the origin of the term "redshift" since visible light is shifted toward the red end of the spectrum. With the numbers given,

$$\frac{v}{c} \approx \frac{\Delta\lambda}{\lambda} = \frac{20\,\mathrm{nm}}{397\,\mathrm{nm}} \approx 0.050 \tag{23}$$

5. A particle with electric charge q moves along a straight line in a uniform electric field $\vec{\mathbf{E}}$ with a speed of v. The electric force exerted on the charge is $q\vec{\mathbf{E}}$. The motion and the electric field are both in the x direction. Show that the acceleration of the particle in the x direction is given by

$$a = \frac{du}{dt} = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2} \tag{24}$$

As with the first problem, we need to start out with a proper relativistic definition of force, which we can equate to the (already relativistically-correct) electric force:

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} = \frac{d}{dt}\left(\gamma m\vec{\mathbf{v}}\right) = m\frac{d}{dt}\left(\gamma\vec{\mathbf{v}}\right) = m\gamma\frac{d\vec{\mathbf{v}}}{dt} + m\vec{\mathbf{v}}\frac{d\gamma}{dt} = q\vec{\mathbf{E}} \tag{25}$$

The difference with the first problem is that speed is no longer constant, so we need to grind through a little more math before we can make use of this. First, we can evaluate $d\gamma/dt$.

$$\frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - v^2/c^2\right)^{-1/2} = \left(-\frac{1}{2}\right)\left(-\frac{2v}{c^2}\right)\left(1 - v^2/c^2\right)^{-3/2}\frac{dv}{dt} = \frac{v}{c^2\left(1 - v^2/c^2\right)^{3/2}}\frac{dv}{dt} \tag{26}$$

You didn't forget the dv/dt, right? With this in hand, we can proceed to solve for dv/dt, the quantity we desire. Since the velocity and electric field are in the same direction, the force and velocity are along the same axis and we can drop the vector notation.

$$qE = m\gamma\frac{dv}{dt} + mv\frac{d\gamma}{dt} = m\gamma\frac{dv}{dt} + \frac{mv}{c^2\left(1 - v^2/c^2\right)^{3/2}}\frac{dv}{dt} \tag{27}$$

$$\frac{qE}{m} = \frac{dv}{dt}\left(\gamma + \frac{v}{c^2\left(1 - v^2/c^2\right)^{3/2}}\right) = \frac{dv}{dt}\left(\frac{1}{\left(1 - v^2/c^2\right)^{1/2}} + \frac{v}{c^2\left(1 - v^2/c^2\right)^{3/2}}\right) \tag{28}$$

$$\frac{qE}{m} = \frac{dv}{dt}\left(\frac{c^2\left(1 - v^2/c^2\right) + v}{c^2\left(1 - v^2/c^2\right)^{3/2}}\right) = \frac{dv}{dt}\frac{1}{\left(1 - v^2/c^2\right)^{3/2}} \tag{29}$$

$$\implies \qquad \frac{dv}{dt} = \left(\frac{qE}{m}\right)\left(1 - \frac{v^2}{c^2}\right)^{3/2} \tag{30}$$

6. *Leighton, 1.10* A stick of length L is at rest on one system and is oriented at an angle θ with respect to the x axis. What are the apparent length and orientation angle of this stick as viewed by an observer moving at a speed v with respect to the first system?

Let the reference frame at rest with respect to the stick be the 'unprimed' frame, with the primed frame corresponding to the observer moving at speed v relative to the stick. Since the relative motion is along the (presumed collinear) x and x' axes, the primed observer sees distances along the x' axis as contracted relative to the reference frame of the stick.

In the stick's (unprimed) frame, the horizontal extent of the stick along the x axis is $L_x = L\cos\theta$, while the extent along the y axis is $L_y = L\sin\theta$. For the moving observer, the x dimensions are contracted, but not the y, and thus

$$\begin{aligned} L'_x &= L_x/\gamma = L_x\sqrt{1-\frac{v^2}{c^2}} \\ L'_y &= L_y \end{aligned} \tag{31}$$

The stationary observer sees the stick as having length $L = \sqrt{L_x^2 + L_y^2}$. The moving observer sees the stick as having a length

$$L' = \sqrt{\left(L'_x\right)^2 + \left(L'_y\right)^2} = \sqrt{L_x^2\left(1-\frac{v^2}{c^2}\right) + L_y^2} = L\sqrt{1-\frac{v^2}{c^2}\cos^2\theta} = L\sqrt{\frac{L_x^2}{\gamma^2} + L_y^2} = L\sqrt{\frac{\cos^2\theta}{\gamma^2} + \sin^2\theta} \tag{32}$$

7. *Leighton, 1.15* A particle appears to move with speed u at an angle θ with respect to the x axis in a certain system. At what speed and angle will this particle appear to move in a second system moving with speed v with respect to the first? Why does the answer differ from that of the previous problem?

It is most straightforward to assume that the two systems have their horizontal x axes aligned. This is still quite general, since we are still letting the particle move at an arbitrary angle θ, we may consider it to be a choice of axes and nothing more. Let the first frame, in which the particle moves with speed u at an angle θ be the 'unprimed' frame (x, y), and the second the 'primed' frame (x', y').

Along the x' direction in the primed frame, both perceived time and distance will be altered. Taking only the x' component of the velocity, we consider the particle's motion purely along the direction of relative motion of the two frames, and we may simply use our velocity addition formula. The x

component of the particle's velocity will in the primed frame become

$$u'_x = \frac{u_x - v}{1 - u_x v/c^2} \tag{33}$$

Along the y' direction in the primed frame, since we consider motion of the particle orthogonal to the direction of relative motion of the frames, there is no length contraction. We need only consider time dilation. We derived this case in class, and the proper velocity addition for directions orthogonal to the relative motion leads to

$$u'_y = \frac{u_y}{\gamma\left(1 - u_x v/c^2\right)} \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{34}$$

The particle's speed in the primed frame is then easily calculated:

$$u' = \sqrt{u'_x u'_x + u'_y u'_y} = \left(\frac{u_x - v}{1 - u_x v/c^2}\right)^2 + \left(\frac{u_y}{\gamma\left(1 - u_x v/c^2\right)}\right)^2 \tag{35}$$

$$= \sqrt{\frac{\left(u_x - v\right)^2 + u_y^2/\gamma^2}{\left(1 - u_x v/c^2\right)^2}} = \frac{\sqrt{\left(u_x - v\right)^2 + u_y^2/\gamma^2}}{1 - u_x v/c^2} \tag{36}$$

As a double-check, we can set $\theta = 0$, such that $u_y = 0$, which corresponds to the particle moving along the x axis. Our expression then reduces to the usual one-dimensional velocity addition formula.

The direction of motion in the primed frame is also found readily:

$$\tan\theta' = \frac{u'_y}{u'_x} = \frac{u_y}{\gamma\left(1 - u_x v/c^2\right)} \frac{1 - u_x v/c^2}{u_x - v} = \left(\frac{u_y}{u_x - v}\right)\sqrt{1 - v^2/c^2} \tag{37}$$

8. *Ohanian 36.44* The acceleration of a particle in one reference frame is $a_x = dv_x/dt$, where the particle has an instantaneous velocity v_x in that frame. Consider a reference frame moving with speed V parallel to the positive x axis of the first frame. Show that the acceleration in the second frame is given by

$$a'_x = \frac{dv'_x}{dt'} = a_x \frac{\left(1 - V^2/c^2\right)^{3/2}}{\left(1 - v_x V/c^2\right)^3}$$

First thing: apply some calculus.

$$a'_x = \frac{dv'_x}{dt'} = \frac{dv'_x/dt}{dt'/dt} \tag{38}$$

What good is this? We know v'_x in terms of v_x and v, and we know t' in terms of t, so the two derivatives we need are trivial. Recall the velocity addition formula, applied to the current problem:

$$v'_x = \frac{v_x - v}{1 - v_x v/c^2} \tag{39}$$

We'll also need the Lorentz transformation for the time coordinates:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{40}$$

Here note that γ involves the relative velocity between the two reference frames, v, not the particle's velocity v_x. Thus, γ does not depend on t since v does not. Here x is just the current position of the particle in the unprimed frame; we won't need it since we're differentiating presently. Given these two transformations,

$$\frac{dt'}{dt} = \gamma\left(1 - \frac{vv_x}{c^2}\right) \tag{41}$$

$$\frac{dv'_x}{dt} = \frac{a_x - 0}{1 - vv_x/c^2} + \frac{-\left(v_x - v\right)\left(-a_x v/c^2\right)}{\left(1 - vv_x/c^2\right)^2} = a_x\left[\frac{1 - vv_x/c^2 + \left(v_x - v\right)\left(v/c^2\right)}{\left(1 - vv_x/c^2\right)^2}\right] \tag{42}$$

Thus,

$$\begin{aligned} a'_x = \frac{dv'_x}{dt'} = \frac{dv'_x/dt}{dt'/dt} &= a_x\left[\frac{1 - vv_x/c^2 + \left(v_x - v\right)\left(v/c^2\right)}{\gamma\left(1 - vv_x/c^2\right)^3}\right] \\ &= a_x\left[\frac{1 - v^2/c^2}{\gamma\left(1 - vv_x/c^2\right)^3}\right] = a_x\left[\frac{\left(1 - v^2/c^2\right)^{3/2}}{\left(1 - vv_x/c^2\right)^3}\right] \end{aligned} \tag{43}$$

9. A particle of mass m is subject to a constant force F along the x axis. If it starts from rest at the origin at time $t=0$, find its position x as a function of time, using relativistic dynamics. Recall that Newton's second law in relativistic form is

$$\vec{\mathbf{F}} = \frac{d\vec{\mathbf{p}}}{dt} \qquad \text{with} \qquad \vec{\mathbf{p}} \equiv \frac{m\vec{\mathbf{v}}}{\sqrt{1 - v^2/c^2}} \tag{44}$$

Note the following useful integral:

$$\int \frac{x}{\sqrt{1+ax^2}}\,dx = \frac{1}{a}\sqrt{1+ax^2} + C \tag{45}$$

We have a constant force along the x axis, let it be $\vec{\mathbf{F}} = F_o\hat{\mathbf{x}}$. Using the definition of force, we can find the momentum. Since this is a one-dimensional problem, we may drop the vector notation.

$$F_o = \frac{dp}{dt} \tag{46}$$
$$dp = F_o dt \tag{47}$$
$$\int dp = \int F_o dt \tag{48}$$
$$p = F_o t + (\text{const}) \tag{49}$$

Our boundary condition is that the particle's velocity is zero at $t = 0$, and thus the integration constant must be zero. Using the equation for relativistic momentum, we can find the velocity:

$$p = \gamma m v = F_o t \tag{50}$$
$$F_o t = \frac{mv}{\sqrt{1 - v^2/c^2}} \tag{51}$$
$$m^2v^2 = F_o^2 t^2 \left(1 - \frac{v^2}{c^2}\right) \tag{52}$$
$$v = \sqrt{\frac{F_o^2 t^2}{m^2 + F_o^2 t^2/c^2}} = \frac{F_o t}{\sqrt{m^2 + F_o^2 t^2/c^2}} \tag{53}$$

Now we can integrate velocity to find position:

$$x(t) = \int_0^t v(t)\,dt = \int_0^t \frac{F_o}{m} \frac{t}{\sqrt{1 + (F_o^2/m^2c^2)\,t^2}} = \left(\frac{F_o}{m}\right)\left(m^2c^2/F_o^2\right)\sqrt{1 + \frac{F_o^2 t^2}{m^2c^2}}\Bigg|_0^t \tag{54}$$
$$= \left(\frac{mc^2}{F_o}\right)\left[\sqrt{1 + \frac{F_o^2 t^2}{m^2c^2}} - 1\right] \tag{55}$$

Note that by integrating from 0 to t we have the correct limiting behavior given by our boundary condition, viz., $x(0) = 0$. Had we performed an indefinite integral, we would have had to impose the condition $x(0) = 0$ to find the integration constant, but we would have arrived at the same result.

Also note that the relativistic trajectory for a particle under the influence of a constant force is a *hyperbola*, not a parabola as we expect classically. For low speeds or small forces, the two are approximately equivalent, but for larger speeds/forces the relativistic path asymptotically approaches a straight line, while the classical path simply diverges. We can check that our result agrees with the classical result by checking the limit that F_ot/mc is small (corresponding to small forces, short times before the speed approaches c, or large masses):

$$x(t) = \left(\frac{mc^2}{F_o}\right)\left[\sqrt{1+\frac{F_o^2t^2}{m^2c^2}}-1\right] \approx \left(\frac{mc^2}{F_o}\right)\left[1+\frac{1}{2}\left(\frac{F_o^2t^2}{m^2c^2}\right)-1\right] = \frac{F_ot^2}{2m} \tag{56}$$

This is just what we would have found classically, starting with $F_ot = ma$ and integrating.