University of Alabama
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PH 253 / LeClair Spring 2010

Problem Set 2: Solutions

1. *Leighton, 2.4* As a function of wavelength, Planck's law states that the emitted power of a black body per unit area of emitting surface, per unit wavelength is

$$I(\lambda,T) = \frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}}-1\right]^{-1} \tag{1}$$

That is, $I(\lambda,T)d\lambda$ gives the emitted power per unit area emitted between wavelengths λ and $\lambda+d\lambda$. Show by differentiation that the wavelength λ_m at which $I(\lambda,T)$ is maximum satisfies the relationship

$$\lambda_m T = b \tag{2}$$

where b is a constant. This result is known as *Wien's Displacement Law,* and can be used to determine the temperature of a black body radiator from only the peak emission wavelength. The constant above has a numerical value of $b = 2.9\times10^6$ nm-K. *Note: at some point you will need to solve an equation numerically. Advice will be given in lecture.*

First, we must find $dI/d\lambda$. Strictly, we want $\partial I/\partial\lambda$, since we are presuming constant temperature, but that is only a formal point since T does not depend on λ. For convenience, define the following substitutions:

$$a \equiv 8\pi hc^2 \tag{3}$$
$$b \equiv \frac{hc}{kT} \tag{4}$$

Thus,

$$I(\lambda,T) = \frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T}}-1\right]^{-1} \frac{a}{\lambda^5}\left[e^{\frac{b}{\lambda}}-1\right]^{-1} \tag{5}$$
$$\frac{dI}{d\lambda} = \frac{-5a}{\lambda^6}\frac{1}{e^{\frac{b}{\lambda}}-1} + \frac{-a}{\lambda^5}\left(\frac{1}{e^{\frac{b}{\lambda}}-1}\right)^2\left(\frac{-be^{\frac{b}{\lambda}}}{\lambda^2}\right) = \left(\frac{a}{\lambda^7}\right)\frac{be^{\frac{b}{\lambda}}-5\lambda e^{\frac{b}{\lambda}}+5\lambda}{\left(e^{\frac{b}{\lambda}}-1\right)^2} = 0 \tag{6}$$

Finding the maximum of $I(\lambda, T)$ with respect to λ means setting $dI(\lambda, T)/d\lambda = 0$.[i] The denominator in the equation above is then irrelevant, as is the λ^{-7} prefactor, and we have

$$0 = be^{\frac{b}{\lambda}} - 5\lambda e^{\frac{b}{\lambda}} + 5\lambda \tag{7}$$

$$0 = be^{\frac{b}{\lambda}} + 5\lambda\left(1 - e^{\frac{b}{\lambda}}\right) \tag{8}$$

$$0 = \frac{be^{\frac{b}{\lambda}}}{\lambda\left(e^{\frac{b}{\lambda}} - 1\right)} \tag{9}$$

We can make another substitution to make things easier. Define $x \equiv \frac{b}{\lambda} = \frac{hc}{\lambda kT}$ and simplify:

$$\frac{xe^x}{e^x - 1} - 5 = 0 \tag{10}$$

If we find the root of this equation, we have (after undoing our substitutions) the value of λ for which $I(\lambda, T)$ is maximum. Unfortunately, there is no analytic solution. Using Newton's method or something similar,[ii] we find the root is

$$x = \frac{hc}{\lambda kT} \approx 4.695 \tag{11}$$

Solving for λ, we obtain the desired result:

$$\lambda_{\max} \approx \frac{hc}{4.965kT} \approx \frac{2.898 \times 10^6 \text{ nm} \cdot \text{K}}{T} \tag{12}$$

2. As a function of *frequency*, Planck's law states that the spectral energy density of a black body, the energy per unit volume per unit frequency, is given by

$$u(f, T) = \frac{8\pi hf^3}{c^3}\left[e^{\frac{hf}{k_b T}} - 1\right]^{-1} \tag{13}$$

If you think of a black body as an insulated, perfectly mirrored box with a tiny hot object inside, $u(f, T)$ would give the energy per unit volume of radiation with frequencies between f and $f + df$. Integrating this energy density over all frequencies, one obtains the *total* energy per unit volume

[i] Since we know the curve is concave downward, we won't bother with the second derivative test; we know very well we will find a maximum and not a minimum.

[ii] See the appendix at the end of this document for some C code that does the job.

V. Show that the total emitted power per unit volume is proportional to T^4. Specifically,

$$\frac{U(T)}{V} = \int_0^\infty u(f,T)\,df = \sigma T^4 \tag{14}$$

Here σ is a constant. This result is essentially the *Stefan-Boltzmann law.* The following integral may be useful:

$$\int_0^\infty \frac{x^3}{e^x - 1}\,dx = \zeta(4)\Gamma(4) = \frac{\pi^4}{90} \times 6 = \frac{\pi^4}{15} \tag{15}$$

A clever substitution might be to define a variable $x = hf/k_bT$.

All we need to do is integrate. It will be less messy with the following substitution:

$$x = \frac{hf}{kT} \tag{16}$$

$$\implies \quad f = \frac{xkT}{h} \tag{17}$$

$$\implies \quad df = \frac{kT}{h}\,dx \tag{18}$$

The limits of integration remain the same, 0 and ∞. You did remember the df, right? With these substitutions, we have

$$\frac{U(T)}{V} = \int_0^\infty u(f,T)\,df = \int_0^\infty u(x,t)\frac{kT}{h}\,dx \tag{19}$$

$$= \frac{8\pi\,(kT)^3}{c^3h^2}\int_0^\infty \frac{x^3}{e^x-1}\left(\frac{kT}{h}\right)\,dx = \frac{8\pi\,(kT)^4}{c^3h^3}\int_0^\infty \frac{x^3}{e^x-1}\,dx \tag{20}$$

The integral is now dimensionless (i.e., it has no units), and is in the end just some number. It happens to be $\pi^4/15$, but this has no real physical significance.[iii] We have already established the proportionality with T^4. Plugging in the value of the integral,

$$\frac{U(T)}{V} = \frac{8\pi^5\,(kT)^4}{15c^3h^3} = \frac{4\sigma T^4}{c} \tag{21}$$

Here σ is the Stefan-Boltzmann constant, which is the proportionality between *power* and T^4.

[iii] We provide a derivation in the appendix at the end of this document.

3. *Leighton, 2.8* The wavelength of maximum intensity in the solar spectrum is about 500 nm, as some of you will verify in PH255. Assuming the sun radiates as a black body, compute its surface temperature.

The Wien displacement law from problem 1 is all we need:

$$T = \frac{2.898 \times 10^6\,\text{nm}\cdot\text{K}}{\lambda_{\text{max}}} \approx 5800\,\text{K} \tag{22}$$

4. In the figure below, the measured intensity as a function of wavelength is shown for a 60 W incandescent bulb at various supply voltages V. (You may ignore the smaller secondary peaks at higher wavelengths, they are due to a phosphorescent coating on the inside of the bulb.) A larger version of this plot is on the last page; raw data is available upon request.

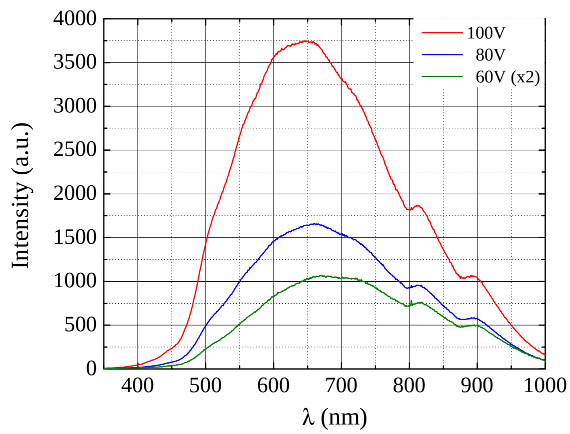

Figure 1: *Spectrum of a* 60 *W soft white incandescent bulb at three different supply voltages, measured in the PH255 lab. Note that the* 60 *V curve has been multiplied by a constant factor!*

(a) Assuming the bulb filament radiates as a perfect black body, the wavelength at which peak intensity occurs should be inversely proportional to temperature, $\lambda_m = b/T$ with $b = 2.9 \times 10^6$ nm-K. Estimate the peak position for each curve. Plot the resulting estimated filament temperature versus the *relative* electrical power supplied to the filament. You may assume the bulb has constant resistance, such that the power supplied to the bulb is proportional to V^2. Do the results make sense?

(b) The total emitted power is proportional to the area under the intensity-wavelength graph. *Roughly estimate* the area under the curves for each voltage. This in turn should be proportional to the bulb temperature to the fourth power, T^4. Plot the estimated area versus for each curve versus T^4 using your temperature estimates from part a. Is the Stefan-Boltzmann law obeyed, within your margin of error?

(c) Is the bulb a reasonable approximation of a black body? You may want to check the melting point of the tungsten filament.

(something to think about, not to turn in) Compare your spectra qualitatively to the solar spectrum, e.g., `http://en.wikipedia.org/wiki/Sunlight`. Can you understand why incandescent bulbs at particular powers are favored for indoor lighting? Why is "color temperature" used to characterize such lighting sources?

In order to calculate the temperature of the bulb from the given spectra, we first assume that the bulb at least roughly behaves a blackbody radiator, and thus Wien's displacement law applies. From the wavelength at which the spectra peaks, we can find the temperature from $\lambda_{\text{peak}} = b/T$. In my case, I find

$$\lambda_{\text{peak}} \approx 648\,\text{nm} \qquad 100\,\text{V} \tag{23}$$

$$\lambda_{\text{peak}} \approx 660\,\text{nm} \qquad 80\,\text{V} \tag{24}$$

$$\lambda_{\text{peak}} \approx 670\,\text{nm} \qquad 60\,\text{V} \tag{25}$$

Assuming a constant bulb resistance, the electrical power supplied is V^2/R. Below, we plot the temperature obtained versus relative power (with 100 V arbitrarily defined as 100% power). The error bars represent the estimated error in peak wavelength determination $\delta\lambda$ propagated to a temperature uncertainty via

$$\left|\frac{\delta\lambda}{\lambda}\right| = \left|\frac{\delta T}{T}\right| \tag{26}$$

This result does make sense: the total radiated power should scale with the total applied electrical power!

For verifying the Stefan-Boltzmann law, we numerically integrated the given curves from the raw data using a simple trapezoid rule in Excel, also estimating the relative error in the standard manner.[iv] Uncertainty in T^4 was propagated according to

$$\left|\frac{\delta T^4}{T}\right| = 4\left|\frac{\delta T}{T}\right| \tag{27}$$

The uncertainty bars on T^4 are not visible on this scale. Within the limits of uncertainty, the Stefan-Boltzmann law is obeyed.

The bulb is *not* an ideal blackbody, for two very simple reasons: first, it is not black, and thus does not absorb all incident radiation; second, the estimated temperature is well above the melting

[iv] See `http://en.wikipedia.org/wiki/Trapezoidal_rule#Error_analysis`

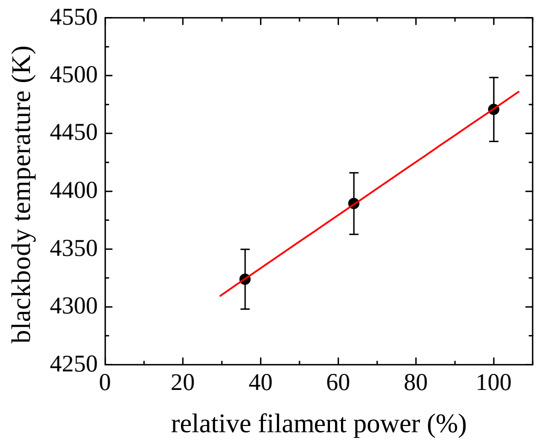

Figure 2: *Bulb temperature obtained from the Wien displacement law versus electrical power, assuming a constant bulb resistance. The red line is a best-fit using a weighted linear regression.*

point of the tungsten filament! One can come up with many other reasons, but either one of these two is sufficient ...

As for the last part, you may find `http://en.wikipedia.org/wiki/Color_temperature` interesting.

5. *Frank, 20.16* Compute the ratio of the increase of intensity of black-body radiation at a wavelength of 641 nm for an increase of temperature from 1200 to 1500 K.

Intensity versus wavelength is quoted in problem 1. The ratio of intensities for a given wavelength $\lambda = 641\,\mathrm{nm}$ at temperatures $T_1 = 1500\,\mathrm{K}$ and $T_2 = 1200\,\mathrm{K}$ is then

$$\frac{I(\lambda, T_1)}{I(\lambda, T_2)} = \frac{\frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T_1}} - 1\right]^{-1}}{\frac{8\pi hc^2}{\lambda^5}\left[e^{\frac{hc}{\lambda k_b T_2}} - 1\right]^{-1}} = \frac{e^{\frac{hc}{\lambda k_b T_1}} - 1}{e^{\frac{hc}{\lambda k_b T_2}} - 1} \approx 42 \tag{28}$$

As you can see, it is much more clever to solve the problem symbolically before using the numbers given. In the opposite case, one ends up performing a great deal of unnecessary calculations.

6. An accelerating charge loses electromagnetic energy at a rate of

$$\mathscr{P} = \frac{\Delta E}{\Delta t} = -\frac{2k_e q^2 a^2}{3c^3}$$

where k_e is Coulomb's constant, q is the charge of the particle, c is the speed of light, and a is the

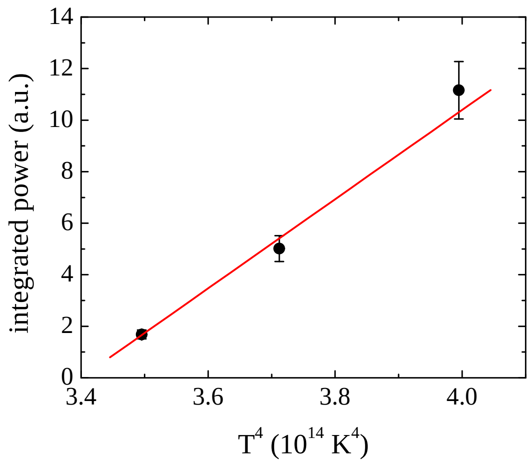

Figure 3: *Total radiated power, determined by finding the area under the $I(\lambda)$ curve, versus estimated temperature to the fourth power. The red line is a best-fit using a weighted linear regression.*

acceleration of the charge. Assume that an electron is one Bohr radius ($a_0 = 0.053\,\text{nm}$) from the center of a Hydrogen atom, with the proton stationary. **(a)** Find the acceleration of the electron (hint: circular path). **(b)** Calculate the kinetic energy of the electron and determine within an order of magnitude how long it will take the electron to loose all of its energy, assuming a constant acceleration as found in part a. Be sure to point out whether you need to consider relativistic effects or not (hint: how big is v/c if you ignore relativity?).

The electron circulating around the proton has only one relevant force, the Coulomb interaction with the proton. This force must provide the centripetal force if the electron is to remain on a circular path:

$$qE = \frac{k_e q^2}{r^2} = ma_{\text{cent}}\frac{mv^2}{r} \qquad \Longrightarrow a_{\text{cent}} = \frac{kq^2}{mr} \sim 9 \times 10^{22}\,\text{m/s}^2 \tag{29}$$

Given that $a_{\text{cent}} = v^2/r$, we also readily find the velocity:

$$v = \sqrt{a_{\text{cent}} r} = \sqrt{\frac{k_e q^2}{mr}} \approx 2 \times 10^6\,\text{m/s} \tag{30}$$

Since $v \sim 0.01c$, we are justified in not using relativistic corrections. The kinetic energy is then simply

$$K = \frac{1}{2}mv^2 = \frac{k_e q^2}{2r} \approx 2.17 \times 10^{-18}\,\text{J} \approx 13.6\,\text{eV} \tag{31}$$

Basically, we have just reproduced the lowest energy level of the hydrogen atom - also known as the ionization energy.

How long is this classical atom stable? We should remember at this point that power is energy per unit time. The power in this case means how much energy we are loosing per unit time, hence the negative sign. What we want to find is how long it will take for the electron to loose *all* of its energy, the whole kinetic energy we just calculated. If, just to obtain an order of magnitude estimate, we assume that the rate of energy loss is constant,

$$\begin{aligned} \mathscr{P} &= \frac{2k_e q^2 a^2}{3c^3} = \frac{2k_e^3 q^6}{3m^2 c^3 r^4} = \frac{\Delta E}{\Delta t} = \frac{K}{\Delta t} \\ \Longrightarrow \quad \Delta t &= \frac{K}{\mathscr{P}} = \frac{3m^2 c^3 r^3}{4k_e^2 e^4} \sim 5 \times 10^{-11}\,\mathrm{s} \end{aligned}$$

Thus, if we can calculate the power - the rate of energy loss - using the (now) known acceleration a and various fundamental constants, we can use the kinetic energy to find out how long it takes the electron to loose all of its energy.

Of course, there are many *many* problems with this analysis. First, the whole orbiting electron model is a kludge of sorts, we know it to be deeply flawed (though still useful). Second, it is perhaps silly to think that the electron looses energy at a constant rate as it continues on its death spiral. Finally, we will soon know better – atoms are indeed stable, and the answer lies in the wave-like nature of matter and the uncertainty principle.

7. Assuming that the human body has a surface area of 2 square meters and radiates like a black body at a temperature of 35°C, calculate the rate at which it loses heat in surroundings that have a temperature of 15°C.

According to the Stefan-Boltzmann law, a body at temperature T of emitting area A radiates a total power P given by

$$P = \sigma A T^4 \tag{32}$$

where σ is the Stefan-Boltzmann constant. How do we calculate the rate of heat loss? Heat is just another form of energy, and if no work is being done on our system of the human plus its surroundings, the change in heat Q is just the change in (internal) energy ΔU. The rate at which energy changes is power. Thus, we need only balance the power lost by the human due to its radiating at a temperature T_h against the power *gained* by absorbing power from its surroundings at a temperature T_s. Or, the net rate of energy loss, the net power, is just power in minus power out. Given that the human both emits and absorbs power over the same area A,

$$P_{\text{net}} = P_h - P_s = \sigma A T_h^4 - \sigma A T_s^4 = \sigma A \left(T_h^4 - T_s^4\right) \tag{33}$$

With the numbers given and $\sigma = 5.67 \times 10^{-8}\,\text{J/m}^2\,\text{s}\,\text{K}^4$, we find $P_{\text{net}} \approx 240\,\text{W}$.

Appendix: Newton's method in C

```c
#include <stdio.h>
#include <math.h>

double newton(double x_0, double tol, int max_iters, int* iters_p, int* converged_p);
double f(double x);
double f_prime(double x);

int main() {
        double x_0;       /* Initial guess                 */
        double x;         /* Approximate solution          */
        double tol;       /* Maximum error                 */
        int    max_iters; /* Maximum number of iterations */
        int    iters;     /* Actual number of iterations  */
        int    converged; /* Whether iteration converged  */

        printf("Enter x_0, tol, and max_iters\n");
        scanf("%lf %lf %d", &x_0, &tol, &max_iters);

        x = newton(x_0, tol, max_iters, &iters, &converged);

        if (converged) {
                printf("Newton algorithm converged after %d steps.\n",
                        iters);
                printf("The approximate solution is %19.16e\n", x);
                printf("f(%19.16e) = %19.16e\n", x, f(x));
        } else {
                printf("Newton algorithm didn't converge after %d steps.\n",
                        iters);
                printf("The final estimate was %19.16e\n", x);
                printf("f(%19.16e) = %19.16e\n", x, f(x));
        }

        return 0;
}  /* main */


double newton(double x_0, double tol, int max_iters,
                        int* iters_p, int* converged_p) {
        double x = x_0;
        double x_prev;
        int    iter = 0;

        do {
```

```
            iter++;
            x_prev = x;
            x = x_prev - f(x_prev)/f_prime(x_prev);
    } while (fabs(x - x_prev) > tol && iter < max_iters);

    if (fabs(x - x_prev) <= tol)
            *converged_p = 1;
    else
            *converged_p = 0;
    *iters_p = iter;

    return x;
}  /* newton algorithm */


double f(double x) {
    return x*exp(x)/(exp(x)-1.0)-5;
}  /* f */

double f_prime(double x) {
    return (exp(2.0*x)-(x+1.0)*exp(x))/pow(exp(x)-1.0,2.0); //the derivative
}  /* f_prime */
```

Appendix: Evaluating $\int_0^\infty x^3\,dx/\left(e^x-1\right)$

Pathologically, the best way to calculate the integral

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx \tag{34}$$

is to calculate a more general case and reduce it to the answer we require. Take the following integral

$$\int_0^\infty \frac{x^n}{e^x-1}\,dx = \int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}\,dx \tag{35}$$

The denominator is always less than one, and is in fact the sum of a geometric series with common multiplier e^{-x}:

$$\frac{1}{1-e^{-x}} = \sum_{k=0}^{\infty} e^{-kx} \tag{36}$$

If we substitute in this series, our integral becomes

$$\int_0^\infty x^n e^{-x} \sum_{k=0}^{\infty} e^{-kx}\, dx \tag{37}$$

We can bring the factor e^{-x} inside our summation, which only shifts the lower limit of the sum from 0 to 1, leaving:

$$\int_0^\infty x^n \sum_{k=1}^{\infty} e^{-kx}\, dx \tag{38}$$

Now make a change of variables $u = kx$, meaning

$$x^n = \frac{u^n}{k^n} \tag{39}$$

$$dx = \frac{du}{k} \tag{40}$$

With this change of variables, our integral is:

$$\int_0^\infty \frac{u^n}{k^n} \sum_{k=1}^{\infty} e^u \frac{du}{k} = \int_0^\infty u^n \sum_{k=1}^{\infty} e^u \frac{du}{k^{n+1}} \tag{41}$$

Each term in the sum represents an integral over u, all of which are convergent. This means we can interchange the order of summation and integration:

$$\sum_{k=1}^{\infty} \frac{1}{k+1} \int_0^\infty u^n e^{-u}\, du \tag{42}$$

The integral on the right side is the definition of the Gamma function $\Gamma(n+1)$, while the summation is then the definition of the Riemann zeta function $\zeta(n+1)$. Thus,

$$\int_0^\infty \frac{x^n}{e^x - 1}\, dx = \zeta(n+1)\Gamma(n+1) \tag{43}$$

With $n = 3$,

$$\zeta(n+1) = \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{44}$$

$$\Gamma(n+1) = n! = 3! = 6 \tag{45}$$

And finally,

$$\int\limits_0^\infty \frac{x^n}{e^x - 1}\, dx = \zeta(n+1)\Gamma(n+1) = \frac{\pi^4}{15} \tag{46}$$