University of Alabama
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Problem Set 3: Solutions

1. The energy required to break one O=O bond in ozone (O_3, O=O=O) is about 500 kJ/mol. What is the maximum wavelength of the photon that has enough energy to photo-dissociate ozone by breaking one of the O=O bonds?

$$O_3 \xrightarrow{hf} O + O_2$$

Note Avagadro's number is $N_A = 6.02 \times 10^{23}$ things/mol.

If we are to break the double oxygen bond in ozone, we need to supply a photon with an energy greater or equal to the bond energy. Two adjacent oxygens are ultimately bonded together because they gain $\Delta E = 500\,\mathrm{kJ/mol}$ worth of energy between them to stay that way - if we want to coax them apart and break the bond, we need to supply that much energy with an incident photon. If we can convert 500 kJ/mol to an energy per bond in, say, electron volts, we can find out what sort of photon has enough energy to break the bond. To do this, we must create an unholy alliance of chemistry and physics units:

$$\begin{aligned}
\Delta E &= 500 \times 10^3 \,\frac{\mathrm{J}}{\mathrm{mol}} \cdot \frac{1\,\mathrm{mol}}{6.02 \times 10^{23}\,\mathrm{bonds}} \\
&= 8.30 \times 10^{-19}\,\frac{\mathrm{J}}{\mathrm{bond}} \cdot \frac{1\,\mathrm{eV}}{1.60 \times 10^{-19}\,\mathrm{J}} \\
&= 5.18\,\frac{\mathrm{eV}}{\mathrm{bond}}
\end{aligned}$$

Now we are getting somewhere! It takes just over 5 electron volts per bond to break an oxygen double bond in an ozone molecule. An incident photon with at least this much energy can be absorbed by one of the oxygen atoms, which will then have enough energy to leave its bound state and break the bond. Thus, to break a single bond:

$$\begin{aligned}
E_{\text{photon}} &= \frac{hc}{\lambda} = 5.18\,\mathrm{eV} \\
\Longrightarrow \quad \lambda &= \frac{hc}{5.18\,\mathrm{eV}} \\
&= \frac{1.24 \times 10^{-6}\,\mathrm{eV \cdot m}}{5.18\,\mathrm{eV}} \\
&= 2.39 \times 10^{-7}\,\mathrm{m} = 239\,\mathrm{nm}
\end{aligned}$$

Here we used our handy relationship from the last problem - $hc = 1.24 \times 10^{-6}\,\text{eV} \cdot \text{m}$. A photon of wavelength 239 nm or lower will break up an ozone molecule, which is well into the ultraviolet (UV). This is one way the ozone layer protects us - it absorbs harmful UV radiation and prevents it from reaching the earth's surface.

2. *Park 1.2* Show that it is impossible for a photon striking a free electron to be absorbed and not scattered.

All we really need to do is conserve energy and momentum for photon absorption by a stationary, free electron and show that something impossible is implied. Before the collision, we have a photon of energy hf and momentum h/λ and an electron with rest energy mc^2. Afterward, we have an electron of energy $(\gamma - 1) + mc^2 = \sqrt{p^2c^2 + m^2c^4}$ (i.e., the afterward the electron has acquired kinetic energy, but retains its rest energy) and momentum $p_e = \gamma mv$. Momentum conservation dictates that the absorbed photon's entire momentum be transferred to the electron, which means it must continue along the same line that the incident photon traveled. This makes the problem one dimensional, which is nice.

Enforcing conservation of energy and momentum, we have:

$$\begin{aligned}
(\text{initial}) &= (\text{final}) && (1)\\
hf + mc^2 &= \sqrt{p^2c^2 + m^2c^4} \qquad \text{energy conservation variant 1} && (2)\\
hf + mc^2 &= (\gamma - 1)\, mc^2 \qquad \text{energy conservation variant 2} && (3)\\
\frac{h}{\lambda} &= p_e = \gamma mv \qquad \text{momentum conservation} && (4)
\end{aligned}$$

From this point on, we can approach the problem in two ways, using either expression for the electron's energy. We'll do both, just to give you the idea. First, we use conservation of momentum to put the electron momentum in terms of the photon frequency:

$$\frac{h}{\lambda} = p_e \qquad \Longrightarrow \qquad \frac{hc}{\lambda} = hf = p_e c \tag{5}$$

Now substitute that in the first energy conservation equation to eliminate p_e, square both sides, and collect terms:

$$\begin{aligned}
\left(hf + mc^2\right)^2 &= \left(\sqrt{p^2c^2 + m^2c^4}\right)^2 = \left(\sqrt{h^2f^2 + m^2c^4}\right)^2 && (6)\\
h^2f^2 + 2hfmc^2 + m^2c^4 &= h^2f^2 + m^2c^4 && (7)\\
2hfmc^2 = 0 \qquad &\Longrightarrow \qquad f = 0 \qquad \Longrightarrow \qquad p_e = v = 0 && (8)
\end{aligned}$$

Thus, we conclude that the only way a photon can be absorbed by the stationary electron is if its frequency is zero, *i.e., if there is no photon to begin with!* Clearly, this is silly.

We can also use the second variant of the conservation of energy equation along with momentum conservation to come to an equally ridiculous conclusion:

$$hf = \frac{hc}{\lambda} = (\gamma - 1)\, mc^2 \qquad \text{energy conservation variant 2} \tag{9}$$

$$\frac{h}{\lambda} = \gamma mv \qquad \text{or} \qquad \frac{hc}{\lambda} = \gamma mvc \qquad \text{momentum conservation} \tag{10}$$

$$\Longrightarrow \qquad \gamma mvc = (\gamma - 1)\, mc^2 \tag{11}$$

$$(\gamma - 1)\, c = \gamma v \tag{12}$$

$$\frac{\gamma - 1}{\gamma} = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \text{(definition of } \gamma) \tag{13}$$

$$\left(\frac{\gamma - 1}{\gamma}\right)^2 = 1 - \frac{1}{\gamma^2} \tag{14}$$

$$\gamma^2 - 2\gamma + 1 = \gamma^2 - 1 \tag{15}$$

$$\gamma = 1 \qquad \Longrightarrow \qquad v = 0 \tag{16}$$

Again, we find an electron recoil velocity of zero, implying zero incident photon frequency, which means there is no photon in the first place! Conclusion: stationary electrons *cannot* absorb photons, but they can Compton scatter them.

3. *Park 1.3* What is the expected recoil velocity of a sodium atom which at rest emits a quantum of its $\lambda = 589.0\,\text{nm}$ radiation?

We need only conservation of momentum. Initially, the sodium atom of mass m is at rest. After the photon emission, the photon carries away momentum $p = h/\lambda$, and conservation of momentum dictates that the sodium atom have equal and opposite momentum $-p = \gamma mv$:

$$0 = \gamma mv - \frac{h}{\lambda} \tag{17}$$

$$\gamma v = \frac{v}{\sqrt{1 - v^2/c^2}} = \frac{h}{m\lambda} \tag{18}$$

$$\frac{h^2}{m^2\lambda^2} = \frac{v^2}{1 - v^2/c^2} \tag{19}$$

$$v^2 = \left(\frac{h^2}{m^2\lambda^2}\right)\left(1 - \frac{v^2}{c^2}\right) \tag{20}$$

$$v = \pm\left(\frac{h}{m\lambda}\right)\left(1 + \frac{h^2}{m^2c^2\lambda^2}\right)^{-1/2} \tag{21}$$

The atomic mass of sodium is $m = 22.99\,\text{u} = 3.82 \times 10^{-26}\,\text{kg}$, leading to $|v| \approx 0.03\,\text{m/s}$. With this small velocity, we really did not require relativity. Using $p \approx mv$ for the sodium atom's momentum, we find

$$v \approx \frac{h}{m\lambda} \tag{22}$$

which is consistent with a Taylor expansion of our relativistic result for $v \ll c$. Incidentally, another way to determine if relativity is really required is to compare the rest energy of the sodium atom and photon. If the latter is relatively small, relativity is not required.

$$\frac{hf}{mc^2} \gg 1 \qquad \Longrightarrow \qquad \frac{h}{\lambda} = p_{\text{photon}} \ll mc \tag{23}$$

If the photon has a negligible fraction of the atom's rest energy, or equivalently its momentum is small compared to mc, the relativistic correction is negligible.

Another thing to think about: if the photon carries away energy, the sodium atom has also effectively lost a mass $\Delta m = hf/c^2$ owing to mass-energy equivalence. This mass is negligibly small in most cases, but we will come back to this point when we consider nuclear reactions.

4. *Ohanian 37.48* Suppose that a photon is "Compton scattered" from a proton instead of an electron. What is the maximum wavelength shift in this case?

The only difference from "normal" Compton scattering is that the proton is heavier. We simply replace the electron mass in the Compton wavelength shift equation with the proton mass, and note that the maximum shift is at $\theta = \pi$:

$$\Delta\lambda_{\text{max}} = \frac{h}{m_p c} \approx 2.64 \times 10^{-15}\,\text{m} = 2.64\,\text{fm} \tag{24}$$

Fantastically small. This is roughly the size attributed to a small atomic nucleus, since the Compton wavelength sets the scale above which the nucleus can be localized in a particle-like sense.

5. The Compton shift in wavelength $\Delta\lambda$ is independent of the incident photon energy $E_i = hf_i$. However, the Compton shift in *energy*, $\Delta E = E_f - E_i$ is strongly dependent on E_i. Find the expression for ΔE. Compute the fractional shift in energy for a 10 keV photon and a 10 MeV photon, assuming a scattering angle of 90°.

The energy shift is easily found from the Compton formula with the substitution $\lambda = hc/E$:

$$\lambda_f - \lambda_i = \frac{hc}{E_f} - \frac{hc}{E_i} = \frac{h}{mc}\left(1-\cos\theta\right) \tag{25}$$

$$\frac{cE_i - cE_f}{E_iE_f} = \frac{1-\cos\theta}{mc} \tag{26}$$

$$\Delta E = E_f - E_i = \left(\frac{E_iE_f}{mc^2}\right)\left(1-\cos\theta\right) \tag{27}$$

$$\frac{\Delta E}{E_i} = \left(\frac{E_f}{mc^2}\right)\left(1-\cos\theta\right) \tag{28}$$

Thus, the fractional energy shift is governed by the photon energy relative to the electron's rest mass, as we might expect. In principle, this is enough: one can plug in the numbers given for E_i and θ, solve for E_f, and then calculate $\Delta E/E_i$ as requested. This is, however, inelegant. One should really solve for the fractional energy change symbolically, being both more elegant and enlightening in the end. Start by dividing both sides of the equation above by E_i to isolate E_f:

$$\frac{E_f}{E_i} + 1 = \frac{E_f}{mc^2}\left(1-\cos\theta\right) \tag{29}$$

$$1 = E_f\left[\frac{1}{E_i} + \frac{1}{mc^2}\left(1-\cos\theta\right)\right] \tag{30}$$

$$E_f = \frac{1}{1/E_i + \left(1-\cos\theta\right)/mc^2} = \frac{mc^2E_i}{mc^2 + E_i\left(1-\cos\theta\right)} \tag{31}$$

Now plug that in to the expression for ΔE we arrived at earlier:

$$\frac{\Delta E}{E_i} = \left(\frac{1}{mc^2}\right)\left(\frac{mc^2E_i}{mc^2 + E_i\left(1-\cos\theta\right)}\right)\left(1-\cos\theta\right) \tag{32}$$

$$\frac{\Delta E}{E_i} = \frac{E_i\left(1-\cos\theta\right)}{mc^2 + E_i\left(1-\cos\theta\right)} = \frac{\frac{E_i}{mc^2}\left(1-\cos\theta\right)}{1 + \frac{E_i}{mc^2}\left(1-\cos\theta\right)} \tag{33}$$

This is even more clear (hopefully): Compton scattering is strongly energy-dependent, and the relevant energy scale is set by the ratio of the incident photon energy to the rest energy of the electron, E_i/mc^2. If this ratio is large, the fractional shift in energy is large, and if this ratio is small, the fractional shift in energy becomes negligible. Only when the incident photon energy is an appreciable fraction of the electron's rest energy is Compton scattering significant. The numerical values required can be found most easily by noting that the electron's rest energy is $mc^2 = 511\,\text{keV}$, which means we don't need to convert the photon energy to joules. One should find:

$$\frac{\Delta E}{E_i} \approx 0.02 \qquad 10\,\text{keV incident photon, } \theta{=}90^\circ \tag{34}$$

$$\frac{\Delta E}{E_i} \approx 0.95 \qquad 10\,\text{MeV incident photon, } \theta{=}90^\circ \tag{35}$$

Consistent with our symbolic solution, for the 10 keV photon the energy shift is negligible, while for the 10 MeV photon it is extremely large. Conversely, this means that the electron acquires a much more significant kinetic energy after scattering from a 10 MeV photon compared to a 10 keV photon.

6. Show that the relation between the directions of motion of the scattered photon and the recoiling electron in Compton scattering is

$$\frac{1}{\tan\left(\theta/2\right)} = \left(1 + \frac{hf_i}{m_e c^2}\right)\tan\varphi \tag{36}$$

Let the electron's recoil angle be φ and the scattered (exiting) photon's angle be θ. Conservation of momentum gets us started. The initial photon momentum is h/λ_i, the final photon momentum is h/λ_f, and the electron's momentum we will simply denote p_e.

$$p_e \sin\varphi = p_f \sin\theta \tag{37}$$

$$p_e \cos\varphi + p_f \cos\theta = p_i \tag{38}$$

We can rearrange the second equation to isolate $p_e \cos\varphi$:

$$p_e \cos\varphi = p_i - p_f \cos\theta \tag{39}$$

Now we can divide Eq. 37 by Eq. 39 to come up with an expression for $\tan\varphi$:

$$\tan\varphi = \frac{p_f \sin\theta}{p_i - p_f\cos\theta} = \frac{\sin\theta}{p_i/p_f - \cos\theta} \tag{40}$$

We now need a substitution for p_i/p_f to eliminate p_f. For this, we can use the Compton equation, which we can rearrange to yield $\lambda_f/\lambda_i{=}p_i/p_f$ in terms of λ_i alone, noting $p{=}h/\lambda$.

$$\lambda_f - \lambda_i = \frac{h}{mc}\left(1-\cos\theta\right) \tag{41}$$

$$\frac{\lambda_f}{\lambda_i} = \frac{p_i}{p_f} = 1 + \frac{h}{mc\lambda_i}\left(1-\cos\theta\right) = 1 + \frac{hf_i}{mc^2}\left(1-\cos\theta\right) \tag{42}$$

For the last line, we used the relationship $\lambda f = c$. Substituting this in Eq. 40, we eliminate p_i and p_f in favor of f_i alone, which we need in our final expression.

$$\tan\varphi = \frac{\sin\theta}{p_i/p_f - \cos\theta} = \frac{\sin\theta}{1 + \frac{hf_i}{mc^2}\left(1-\cos\theta\right) - \cos\theta} = \frac{\sin\theta}{\left(1 + \frac{hf_i}{mc^2}\right)\left(1-\cos\theta\right)} \tag{43}$$

With the aid of a rather obscure trigonometric identity, we can obtain the desired result. Specifically:

$$\frac{1-\cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right) \tag{44}$$

Using this in Eq. 43,

$$\left(1 + \frac{hf_i}{mc^2}\right)\tan\varphi = \frac{1}{\tan\left(\theta/2\right)} \tag{45}$$

If we define a dimensionless energy/momentum $\alpha_i = \frac{hf_i}{mc^2} = \frac{h}{mc\lambda_i} = \frac{p_i}{mc}$ the result is somewhat simpler, as is the Compton equation:

$$\left(1+\alpha_i\right)\tan\varphi = \frac{1}{\tan\left(\theta/2\right)} \tag{46}$$

$$\frac{\alpha_i}{\alpha_f} = 1 + \alpha_i\left(1-\cos\theta\right) \qquad \text{(Compton)} \tag{47}$$

This simplification has utility, because it will allow us to derive the electron energy in a more compact fashion for the last question (see below).

7. *French & Taylor 1.8* A radio station broadcasts at a frequency of 1 MHz with a total radiated power of 5 kW. **(a)** What is the wavelength of this radiation? **(b)** What is the energy (in electron volts) of the individual quanta that compose the radiation? How many photons are emitted per second? Per cycle of oscillation? **(c)** A certain radio receiver must have 2 μW of radiation power incident on its antenna in order to provide an intelligible reception. How many 1 MHz photons does this require per second? Per cycle of oscillation? **(d)** Do your answers for parts (b) and (c) indicate that the granularity of electromagnetic radiation can be neglected in these circumstances?

(a) Radio waves are just light, so knowledge of the frequency gives us the wavelength:

$$\lambda = \frac{c}{f} = 300\,\mathrm{m} \tag{48}$$

(b) The energy of an individual photon is just $hf = 4.1 \times 10^{-9}\,\text{eV} = 6.63 \times 10^{-28}\,\text{J}$. The station's power ($P$) is the energy ($\Delta E$) per unit time ($\Delta t$) emitted, and must just be the energy per photon times the number of photons per unit time. If we call the number of photons per unit time $\Delta N/\Delta t$,

$$P = \frac{\Delta E}{\Delta t} = hf\frac{\Delta N}{\Delta t} \qquad \Longrightarrow \qquad \frac{\Delta N}{\Delta t} = \frac{P}{hf} \approx 7.5 \times 10^{30}\,\text{photons/s} \tag{49}$$

There are 10^6 periods of oscillation per second, so that means that there are approximately 7.5×10^{24} photons/period being emitted.

(c) This is precisely the same as the previous question, except the relevant power is $2\,\mu\text{W}$ instead of 5000 W.

$$\frac{\Delta N}{\Delta t} = \frac{P}{hf} \approx 3.0 \times 10^{21}\,\text{photons/s} \tag{50}$$

Again, there are 10^6 periods of oscillation per second, so there are approximately 3.0×10^{15} photons/period being emitted.

This is certainly enough photons that the granularity of electromagnetic radiation is utterly negligible for everyday power levels such as these.

What *would* the power level have to be for 1 MHz photons to have a noticeable granularity? Roughly speaking, the sampling theorem says that if a function $x(t)$ contains no frequencies higher than B, it is completely determined by sampling at a rate of $1/2B$.[i]. We could say then that the granularity in a signal would be noticeable in this case if the photons were coming at less than 2 per cycle of oscillation. That means

$$\frac{\Delta N}{\Delta t} = \frac{P}{hf} \approx 2\,\text{photons/period} = 2 \times 10^6\,\text{photons/sec} \tag{51}$$

With the given photon frequency of 1 MHz, we find $P \sim 10^{-21}\,\text{W}$, a negligible amount of power. For photons of visible light, in the 10^{15} Hz range, the power is $\sim 10^{-12}\,\text{W}$, which is close to the limit of human vision. With dark-adapted scotopic vision, we detect about $8 \times 10^{-11}\,\text{W/m}^2$ of green light (550 nm), which means down to around $\sim 10^2 - 10^3$ photons/s for an average-sized eye. Just about enough to notice the granularity, but not quite.[ii]

[i] `http://en.wikipedia.org/wiki/Nyquist-Shannon_sampling_theorem`

[ii] Actually, it is more complicated than this. The sensors in the eye are capable of detecting single photons, but our neural hardware filters the incoming signals to smooth out this granularity. If it didn't, we would be too distracted by the granularity in low light. See `http://math.ucr.edu/home/baez/physics/Quantum/see_a_photon.html` for a nice discussion.

8. *French & Taylor 1.11* The clean surface of sodium metal (in vacuum) is illuminated with monochromatic light of various wavelengths and the retarding potentials required to stop the most energetic photoelectrons are observed as follows:

Wavelength (nm)	253.6	283.0	303.9	330.2	366.3	435.8
Stopping potential (V)	2.60	2.11	1.81	1.47	1.10	0.57

Plot these data in such a way as to show that they lie (approximately) on a straight line as predicted by the photoelectric equation, and obtain a value for h and the work function of sodium in electron volts.

The plot we require is one of stopping potential versus frequency. The slope then yields h/e, and the y intercept the work function.

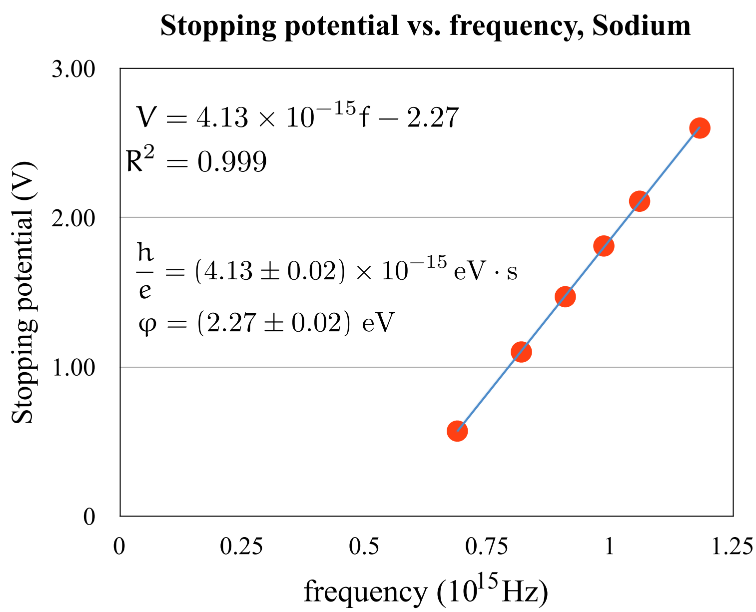

Figure 1: *Stopping potential versus incident photon frequency for sodium metal. Linear regression gives* $h/e = (4.13 \pm 0.02) \times 10^{15}\,eV \cdot s$ *and* $\varphi = 2.27 \pm 0.02\,eV$ *with a correlation coefficient* $R^2 = 0.999$.

9. *Ohanian 37.51* What is the maximum energy that a free electron (initially stationary) can acquire in a collision with a photon of energy 4 keV?

We can exploit our results from problem 6 to come up with a relatively simple expression for the electron energy. The Compton equation, expressed in terms of the dimensionless energies $\alpha_i = hf_i/mc^2$ and $\alpha_f = hf_f/mc^2$, becomes:

$$\frac{\alpha_i}{\alpha_f} = 1 + \alpha_i\,(1 - \cos\theta) \tag{52}$$

Conservation of energy dictates that the electron energy E_e must simply be the difference between incident and exiting photon energies:

$$E_e = E_i - E_f = \frac{hc}{\lambda_i} - \frac{hc}{\lambda_f} = \alpha_i mc^2 - \alpha_f mc^2 = \alpha_i mc^2\left(1 - \frac{\alpha_f}{\alpha_i}\right) \tag{53}$$

$$E_e = \alpha_i mc^2\left(1 - \frac{1}{1+\alpha_i\left(1-\cos\theta\right)}\right) = \alpha_i mc^2\left[\frac{1+\alpha_i\left(1-\cos\theta\right)-1}{1+\alpha_i\left(1-\cos\theta\right)}\right] \tag{54}$$

$$E_e = mc^2\left[\frac{\alpha_i^2\left(1-\cos\theta\right)}{1+\alpha_i\left(1-\cos\theta\right)}\right] = hf_i\left[\frac{\alpha_i\left(1-\cos\theta\right)}{1+\alpha_i\left(1-\cos\theta\right)}\right] \tag{55}$$

With sufficient interest, one can go on to show two other interesting relationships:

$$E_e = mc^2\left[\frac{2\alpha_i^2}{1+2\alpha_i+\left(1+\alpha_i\right)^2\tan^2\varphi}\right] \tag{56}$$

$$\cos\theta = 1 - \frac{2}{\left(1+\alpha_i\right)^2\tan^2\varphi+1} \tag{57}$$

However, we have no need of these relationships at the moment ... all we really need to do is maximize E_e with respect to θ. One could simply assert the maximum is clearly when $\cos\theta = -1$, i.e., $\theta = \pi$, but this is unsatisfying and perhaps a touch arrogant. We can set $dE/d\theta = 0$ to be sure:

$$\frac{dE}{d\theta} = \alpha_i^2 mc^2\left[\frac{-\alpha_i\sin\theta}{\left(1+\alpha_i\left(1-\cos\theta\right)\right)^2} + \frac{\sin\theta}{1+\alpha_i\left(1-\cos\theta\right)} + \frac{\alpha_i\sin\theta\cos\theta}{\left(1+\alpha_i\left(1-\cos\theta\right)\right)^2}\right] = 0 \tag{58}$$

$$0 = \sin\theta\left[-\alpha_i + 1 + \alpha_i\left(1-\cos\theta\right) + \alpha_i\cos\theta\right] \tag{59}$$

$$0 = \sin\theta \tag{60}$$

$$\theta = \{0, \pi\} \tag{61}$$

The solution $\theta = 0$ can be discarded, since this corresponds to the photon going right through the electron, an unphysical result. One should also perform the second derivative test to ensure we have found a maximum, but it is tedious and can be verified by a quick plot of $E(\theta)$. At $\theta = \pi$, the maximum energy of the electron thus takes a nicely simple form:

$$E_{\text{max}} = hf\left(\frac{2\alpha_i}{1+2\alpha_i}\right) \approx 62\,\text{eV} \tag{62}$$

For the numerical answer, we noted that $\alpha_i = hf_i/mc^2 = (4\,\text{keV})\,/\,(511\,\text{keV}) \approx 7.8\times10^{-3}$.