UNIVERSITY OF ALABAMA
Department of Physics and Astronomy

PH 253 / LeClair Spring 2010

Problem Set 4

1. *Group velocity of a wave.* For a free relativistic quantum particle moving with speed v, the total energy is $E=hf=\hbar\omega=\sqrt{p^2c^2+m^2c^4}$ and the momentum is $p=h/\lambda=\hbar k=\gamma mv$. For the quantum wave representing the particle, the group speed is $v_g=d\omega/dk$. Prove that the group speed of the wave is the same as the speed of the particle.

2. *Uncertainty in everyday processes.* A woman on a ladder drops small pellets toward a point target on the floor. **(a)** Show that, according to the uncertainty principle, the average miss distance must be at least

$$\Delta x_f = \sqrt{\frac{2\hbar}{m}}\left(\frac{2H}{g}\right)^{1/4} \tag{1}$$

where H is the initial height of each pellet above the floor and m is the mass of each pellet. Assume that the spread in impact points is given by $\Delta x_f=\Delta x_i+(\Delta v_x)\,t$. **(b)** If $H=2.00\,\mathrm{m}$ and $m=0.500\,\mathrm{g}$, what is Δx_f?

3. *Significance of the Compton wavelength.* Show that the speed of a particle having de Broglie wavelength λ and a Compton wavelength $\lambda_c=h/mc$ is

$$v = \frac{c}{\sqrt{1+(\lambda/\lambda_c)^2}} \tag{2}$$

4. *Zero point energy of a harmonic oscillator.* The frequency f of a harmonic oscillator of mass m and elasticity constant k is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{3}$$

The energy of the oscillator is given by

$$E = \frac{p^2}{2m}+\frac{1}{2}kx^2 \tag{4}$$

where p is the system's linear momentum and x is the displacement from its equilibrium position. Use the uncertainty principle, $\Delta x\Delta p\approx\hbar/2$, to express the oscillator's energy E in terms of x and show, by taking the derivative of this function and setting $dE/dx=0$, that the minimum energy of the oscillator (its ground state energy) is $E_{\text{min}}=hf/2$.

5. *Deduction of atomic energy levels.* The following lines appear in the emission spectrum of an element:

λ_1=122 nm	λ_2=102.5 nm	λ_3=97 nm	λ_4=95 nm	λ_5=656 nm
λ_6=486 nm	λ_7=434 nm	λ_8=1880 nm	λ_9=1280 nm	λ_{10}=4050 nm

(a) Does the spectrum exhibit the Ritz combination rule?
(b) Calculate the energy of the photons associated with each spectral line.
(c) Sketch an energy level diagram that is consistent with the element's spectral lines.

6. *Macroscopic quantum matter.* Under usual conditions of temperature and density, the behavior of a gas can be interpreted in terms of classical physics, for example, by Maxwell's kinetic theory (treating the gas as a collection of point particles). However, when the de Broglie wavelength of the atoms is of the order of the mean distance d between them, the classical approach fails. **(a)** Show that the classical approach fails when

$$d \approx \sqrt{\frac{h^2}{mk_bT}} \tag{5}$$

(b) Assuming that the mean distance between the atoms is 1 nm, at what temperature will quantum effects begin to appear in the gas helium?

7. *How big are atoms?* A simple but sophisticated argument holds that the hydrogen atom has its observed size because this size minimizes the total energy of the system. The argument rests on the assumption that the lowest-energy state corresponds to a physical size comparable to a de Broglie wavelength of the electron. Larger size means larger de Broglie wavelength, hence smaller momentum and kinetic energy. In contrast smaller size means lower potential energy, since the potential well is deepest near the proton. The observed size is a compromise between kinetic and potential energies that minimizes the total energy of the system. Develop the argument explicitly, for example as follows:

- **(a)** Write down the classical expression for the total energy of the hydrogen atom with an electron of momentum p in a circular orbit of radius r. Keep kinetic and potential energies separate.
- **(b)** Failure of classical energy minimization. Use the force law to obtain the total energy as a function of radius. What radius corresponds to the lowest possible energy?
- **(c)** For the lowest-energy state, demand that the orbit circumference be one de Broglie wavelength. Obtain an expression for the total energy as a function of radius. Note how a larger

radius decreases the kinetic and increases the potential energy, whereas a smaller radius increases the kinetic and decreases the potential energy.

(d) Take the derivative of the energy versus radius function and find the radius that minimizes the total energy. How large is that radius for a hydrogen atom? For a He^+ ion with one electron?

8. (a) *Early neutron model.* The neutron is an electrically neutral particle with a mass approximately equal to the proton mass. An early model considered the neutron to be an object where the electron is confined inside the proton. Assuming that the proton radius is $r \approx 10^{-15}$ m, estimate the electron's kinetic energy due to Heisenberg uncertainty, and compare it to the neutron rest mass.

(b) *Energy spread of electron beam.* A monochromatic beam of electrons of energy $E = 1\,\mathrm{keV}$ is incident on a shutter that opens for $\Delta t = 1\,\mathrm{ns}$. What is the fractional energy spread $\Delta v/v$ of the electron velocity v after the shutter? (Decide first whether to perform a relativistic or a non-relativistic calculation.)