University of Alabama
Department of Physics and Astronomy

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Problem Set 4 Solutions

1. *Group velocity of a wave.* For a free relativistic quantum particle moving with speed v, the total energy is $E = hf = \hbar\omega = \sqrt{p^2c^2 + m^2c^4}$ and the momentum is $p = h/\lambda = \hbar k = \gamma m v$. For the quantum wave representing the particle, the group speed is $v_g = d\omega/dk$. Prove that the group speed of the wave is the same as the speed of the particle.

We can just brute-force this one. Using the energy equation, we can write ω in terms of k:

$$\omega = \frac{1}{\hbar}\sqrt{\hbar^2k^2c^2 + m^2c^4} \tag{1}$$

$$v_g = \frac{d\omega}{dk} = \frac{1}{\hbar}\frac{\frac{1}{2}\left(2\hbar^2kc^2\right)}{\sqrt{\hbar^2k^2c^2 + m^2c^4}} \tag{2}$$

$$= \frac{\hbar kc^2}{\sqrt{\hbar^2k^2c^2 + m^2c^4}} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \sqrt{\frac{p^2c^4}{p^2c^2 + m^2c^4}} \tag{3}$$

In the last line, we substituted back in $p = \hbar k$. If we use $p = \gamma m v$, we can reduce this expression to the desired result:

$$\frac{d\omega}{dk} = \sqrt{\frac{p^2c^4}{p^2c^2 + m^2c^4}} = \sqrt{\frac{\gamma^2m^2v^2c^4}{\gamma^2m^2v^2c^2 + m^2c^4}} = \sqrt{\frac{\gamma^2c^2v^2}{\gamma^2v^2 + c^2}} \tag{4}$$

$$= \sqrt{\frac{\frac{c^2v^2}{1 - v^2/c^2}}{\frac{v^2}{1 - v^2/c^2} + \frac{c^2\left(1 - v^2/c^2\right)}{1 - v^2/c^2}}} = \sqrt{\frac{c^2v^2}{v^2 + c^2 - v^2}} = \pm v \tag{5}$$

$$\therefore \quad |v_g| = |v| \tag{6}$$

2. *Uncertainty in everyday processes.* A woman on a ladder drops small pellets toward a point target on the floor. **(a)** Show that, according to the uncertainty principle, the average miss distance must be at least

$$\Delta x_f = \sqrt{\frac{2\hbar}{m}}\left(\frac{2H}{g}\right)^{1/4} \tag{7}$$

where H is the initial height of each pellet above the floor and m is the mass of each pellet. Assume that the spread in impact points is given by $\Delta x_f = \Delta x_i + (\Delta v_x)\, t$. **(b)** If $H = 2.00\,\mathrm{m}$ and $m = 0.500\,\mathrm{g}$, what is Δx_f?

Owing to the quantum nature of matter, the initial position of the pellet x_i will have some uncertainty Δx_i. From the uncertainty principle, this implies there must also be an uncertainty in the initial momentum in the x direction Δp_{ix}, and therefore the initial x component of the velocity as well, Δv_{ix}. This random initial velocity will really have both a horizontal component Δv_{ix} and a vertical component Δv_{iy}. However, since the particle will quickly acquire a sizable vertical speed, the vertical uncertainty will be negligible, and it will not affect the horizontal position at which the particle lands. Thus, we need only consider the initial random horizontal speed Δv_{ix} which will lead to a scatter in the horizontal position of the particle.

Classically, the particle has a purely vertical velocity, so the *total* velocity in the horizontal direction is only that which comes from uncertainty. After a time t, this will cause the particle to have moved horizontally by $\Delta v_{ix} t$ compared to its expected position. Adding this to the initial uncertainty in position, after a time t our total uncertainty is

$$\Delta x(t) = \Delta x_i + \Delta v_{ix} t \tag{8}$$

If the particle starts from rest at height H, the time to reach the ground is just $t = \sqrt{2H/g}$. Thus, the final uncertainty in the horizontal position at which the particle hits the ground is

$$\Delta x_f = \Delta x_i + \Delta v_{ix} \sqrt{\frac{2H}{g}} \tag{9}$$

From the uncertainty principle, we may relate the *minimum* uncertainty in initial velocity to the minimal uncertainty in initial position:

$$\Delta x_i \Delta p_{ix} = m \Delta x_i \Delta v_{ix} = \frac{\hbar}{2} \qquad \Longrightarrow \qquad \Delta v_{ix} = \frac{\hbar}{2m\Delta x_i} \tag{10}$$

Using this in our expression for the final spread in horizontal position,

$$\Delta x_f = \Delta x_i + \frac{\hbar}{2m\Delta x_i} \sqrt{\frac{2H}{g}} \tag{11}$$

We can find the minimum spread in Δx_f by minimizing with respect to Δx_i, and this will tell us the degree of uncertainty in the final position required by the uncertainty principle – it can be more, but this will give us the minimal value.

$$\frac{d\left(\Delta x_f\right)}{d\left(\Delta x_i\right)} = 1 - \frac{\hbar}{2m\left(\Delta x_i\right)^2}\sqrt{\frac{2H}{g}} \qquad \Longrightarrow \qquad \Delta x_i = \sqrt{\frac{h}{2m}}\left(\frac{2H}{g}\right)^{1/4} \tag{12}$$

Plugging this in to our previous expression and simplifying,

$$\Delta x_f = \sqrt{\frac{h}{2m}}\left(\frac{2H}{g}\right)^{1/4} + \frac{\hbar}{2m}\sqrt{\frac{2m}{\hbar}}\left(\frac{g}{2H}\right)^{1/4}\sqrt{\frac{2H}{g}} = \sqrt{\frac{2\hbar}{m}}\left(\frac{2H}{g}\right)^{1/4} \approx 5\times 10^{-16}\,\mathrm{m} \tag{13}$$

3. *Significance of the Compton wavelength.* Show that the speed of a particle having de Broglie wavelength λ and a Compton wavelength $\lambda_c = h/mc$ is

$$v = \frac{c}{\sqrt{1+\left(\lambda/\lambda_c\right)^2}} \tag{14}$$

There are many ways to go about this one. Here is a short one. Since $\lambda_c = \frac{h}{mc}$, then $\lambda_c c = \frac{h}{m}$. The de Broglie wavelength is thus

$$\lambda = \frac{h}{p} = \frac{h}{\gamma m v} = \frac{h}{m}\frac{1}{\gamma v} = \frac{\lambda_c c}{\gamma v} \tag{15}$$

Squaring both sides and simplifying,

$$\lambda^2 = \lambda_c^2\frac{c^2}{v^2}\left(1-\frac{v^2}{c^2}\right) = \lambda_c^2\left(\frac{c^2}{v^2}-1\right) \tag{16}$$

$$\frac{c^2}{v^2} = 1 + \frac{\lambda^2}{\lambda_c^2} \tag{17}$$

$$\frac{v^2}{c^2} = \frac{1}{1+\frac{\lambda^2}{\lambda_c^2}} \tag{18}$$

$$|v| = \frac{c}{\sqrt{1+\left(\lambda/\lambda_c\right)^2}} \tag{19}$$

4. *Zero point energy of a harmonic oscillator.* The frequency f of a harmonic oscillator of mass m and elasticity constant k is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{20}$$

The energy of the oscillator is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{21}$$

where p is the system's linear momentum and x is the displacement from its equilibrium position. Use the uncertainty principle, $\Delta x \Delta p \approx \hbar/2$, to express the oscillator's energy E in terms of x and show, by taking the derivative of this function and setting $dE/dx = 0$, that the minimum energy of the oscillator (its ground state energy) is $E_{\text{min}} = hf/2$.

The minimum uncertainty in momentum Δp, given an uncertainty Δx in position is given by the uncertainty principle:

$$\Delta x \Delta p = \frac{\hbar}{2} \qquad \Longrightarrow \qquad \Delta p = \frac{\hbar}{2\Delta x} \tag{22}$$

The minimum uncertainty is also then the minimum average value we can expect either variable to take on: $p_{\text{min}} = \Delta p \equiv p$, $x_{\text{min}} = \Delta x \equiv x$. The energy equation may the be rewritten in terms of the minimal x and p:

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}kx^2 = \frac{\hbar^2}{8mx^2} + \frac{1}{2}m\omega^2 x^2 \tag{23}$$

In the last line, we used $\omega = \sqrt{k/m}$, so $k = m\omega^2$. Minimizing the energy with respect to x,

$$\frac{dE}{dx} = \frac{-2\hbar^2}{4mx^3} + m\omega^2 x = 0 \qquad \Longrightarrow \qquad x^2 = \frac{\hbar}{2m\omega} \tag{24}$$

Plugging this back in to the energy equation, we have the minimum energy:

$$E_{\text{min}} = \frac{\hbar^2}{8m}\frac{2m\omega}{\hbar} + \frac{1}{2}m\omega^2 \frac{\hbar}{2m\omega} = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \frac{1}{2}\hbar\omega \tag{25}$$

5. *Deduction of atomic energy levels.* The following lines appear in the emission spectrum of an element:

λ_1=122 nm	λ_2=102.5 nm	λ_3=97 nm	λ_4=95 nm	λ_5=656 nm
λ_6=486 nm	λ_7=434 nm	λ_8=1880 nm	λ_9=1280 nm	λ_{10}=4050 nm

(a) Does the spectrum exhibit the Ritz combination rule?
(b) Calculate the energy of the photons associated with each spectral line.
(c) Sketch an energy level diagram that is consistent with the element's spectral lines.

In fact, the wavelengths correspond to the hydrogen spectrum. Armed with that knowledge, the problem is much simpler. Each given wavelength corresponds to the energy difference between two levels of the hydrogen atom (say, $n=3$ and $n=1$). Besides the direct transition from level 3 to level 1, the electron could also go from 3 to 2 and then 2 to 1, meaning there must be two other photon energies that sum to that of the first photon: $E_{3\rightarrow 1} = E_{3\rightarrow 2} + E_{2\rightarrow 1}$. Using the energy level diagram below[i], it is a simple matter to write down all possible energy level transitions and their energies, and match them to the list.

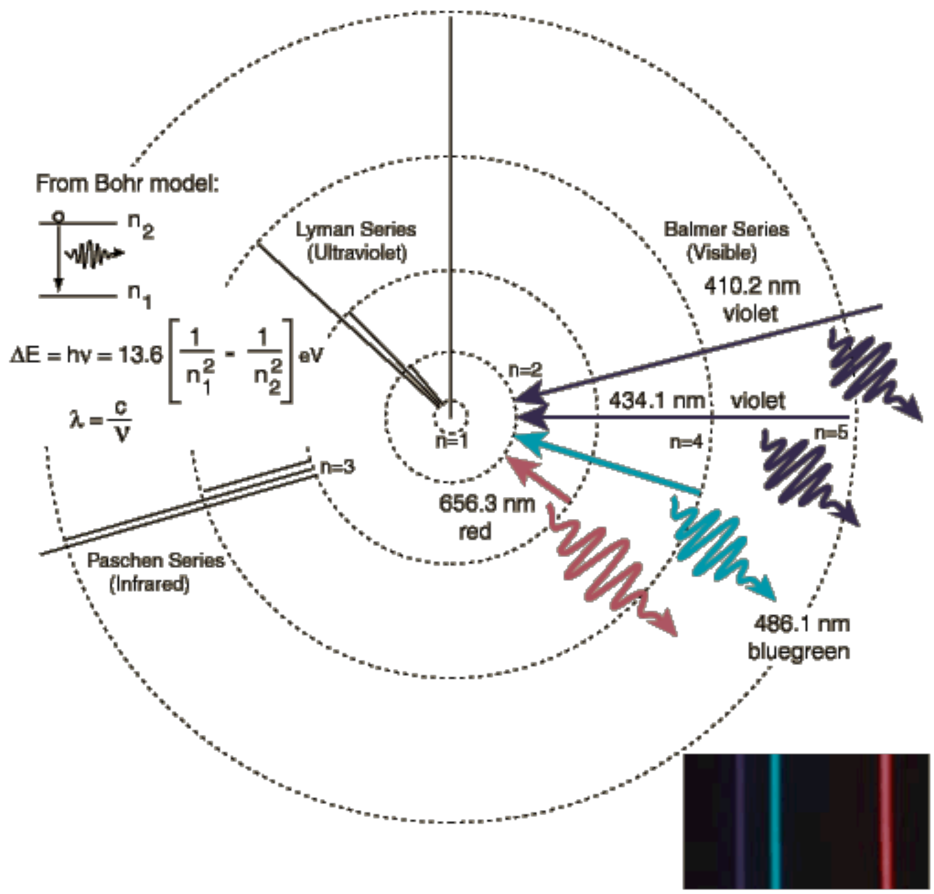

Figure 1: *In the Bohr model of the hydrogen atom, the emission lines correspond to an electron jump from a higher energy level n to a lower n' with the emission of a photon corresponding to the energy difference between the two levels. For the visible Balmer series, $n'=2$ and n runs from 3 upward. From* `http://hyperphysics.phy-astr.gsu.edu/`.

The table below lists the energies corresponding to the given wavelengths, their identification, and their decomposition into sums of other given energies. If no decomposition is listed, it means that this wavelength corresponds to the first in a series, and cannot be decomposed into smaller individual transitions.

If you did't realize the given wavelengths were from the hydrogen spectra, one could also brute-force

[i] From `http://hyperphysics.phy-astr.gsu.edu/hbase/hyde.html`.

level	λ (nm)	f (PHz)	E (eV)	transition	Decompositions
E_0	95	3.158	13.053	Lyman 5→1	E_1+E_9; E_2+E_7; E_3+E_4 $E_2+E_8+E_9$; $E_3+E_5+E_9$; $E_3+E_6+E_7$ $E_3+E_6+E_8+E_9$
E_1	97	3.093	12.784	Lyman 4→1	$E_8+E_6+E_3$; E_5+E_3; E_8+E_2
E_2	102.5	2.927	12.098	Lyman 3→1	E_6+E_3
E_3	122	2.459	10.164	Lyman 2→1	–
E_4	434	0.691	2.857	Balmer 5→2	$E_9+E_8+E_6$; E_9+E_5; E_7+E_6
E_5	486	0.617	2.551	Balmer 4→2	E_8+E_6
E_6	656	0.457	1.890	Balmer 3→2	–
E_7	1280	0.234	0.969	Paschen 5→3	E_9+E_8
E_8	1880	0.160	0.660	Paschen 4→3	–
E_9	4050	0.074	0.306	Brackett 5→4	–

the problem. I wrote a small C program to search for combinations of the energies corresponding to the given wavelengths. Here are the results, which match the predictions of the Bohr model nicely. I ordered the wavelengths given from lowest to highest, meaning the corresponding energies E_0 to E_9 are ordered from highest to lowest. The brute-force search was carried only as far as four terms.

$$
\begin{aligned}
E_0 &= 13.1 & E_1 + E_9 &= 13.1\\
E_0 &= 13.1 & E_2 + E_7 &= 13.1\\
E_0 &= 13.1 & E_3 + E_4 &= 13.0\\
E_1 &= 12.8 & E_2 + E_8 &= 12.8\\
E_1 &= 12.8 & E_3 + E_5 &= 12.7\\
E_2 &= 12.1 & E_3 + E_6 &= 12.1\\
E_4 &= 2.86 & E_5 + E_9 &= 2.86\\
E_4 &= 2.86 & E_6 + E_7 &= 2.86\\
E_5 &= 2.55 & E_6 + E_8 &= 2.55\\
E_7 &= 0.969 & E_8 + E_9 &= 0.966\\
E_0 &= 13.1 & E_2 + E_8 + E_9 &= 13.1\\
E_0 &= 13.1 & E_3 + E_5 + E_9 &= 13.0\\
E_0 &= 13.1 & E_3 + E_6 + E_7 &= 13.0\\
E_1 &= 12.8 & E_3 + E_6 + E_8 &= 12.7\\
E_4 &= 2.86 & E_6 + E_8 + E_9 &= 2.86\\
E_0 &= 12.8 & E_3 + E_6 + E_8 + E_9 &= 13.0
\end{aligned}
$$

Thus, the Ritz combination rule is obeyed, even if one didn't realize the spectrum was for hydrogen.

6. *Macroscopic quantum matter.* Under usual conditions of temperature and density, the behavior of a gas can be interpreted in terms of classical physics, for example, by Maxwell's kinetic theory (treating the gas as a collection of point particles). However, when the de Broglie wavelength of the atoms is of the order of the mean distance d between them, the classical approach fails. **(a)** Show that the classical approach fails when

$$d \approx \sqrt{\frac{h^2}{mk_bT}} \tag{26}$$

(b) Assuming that the mean distance between the atoms is 1 nm, at what temperature will quantum effects begin to appear in the gas helium?

The average speed of a particle in an ideal gas is

$$\langle v \rangle = \sqrt{\frac{3k_BT}{m}} \tag{27}$$

which lets us write the average de Broglie wavelength as

$$\langle \lambda \rangle = \frac{h}{\langle p \rangle} = \frac{h}{m\langle v \rangle} = \frac{h}{\sqrt{3k_BTm}} \tag{28}$$

The classical approach is expected to fail when the spacing between atoms approaches the de Broglie wavelength, or

$$d \approx \frac{h}{\sqrt{3k_BTm}} \sim \frac{h}{\sqrt{k_BTm}} \tag{29}$$

At room temperature, this is about 0.07 nm; at 4.2 K (the boiling point of helium), it is about 0.6 nm. This already hints that quantum effects should arise in helium near its boiling point, and certainly in the liquid. If we assume a mean spacing of 1 nm for the helium atoms, the temperature at which the classical approach breaks down is

$$T = \frac{h^2}{3k\langle \lambda \rangle^2 m} \approx 2\,\mathrm{K} \tag{30}$$

7. *How big are atoms?* A simple but sophisticated argument holds that the hydrogen atom has its observed size because this size minimizes the total energy of the system. The argument rests on the assumption that the lowest-energy state corresponds to a physical size comparable to a de Broglie wavelength of the electron. Larger size means larger de Broglie wavelength, hence smaller momentum and kinetic energy. In contrast smaller size means lower potential energy, since the potential well is deepest near the proton. The observed size is a compromise between kinetic and potential energies that minimizes the total energy of the system. Develop the argument explicitly, for example as follows:

(a) Write down the classical expression for the total energy of the hydrogen atom with an electron of momentum p in a circular orbit of radius r. Keep kinetic and potential energies separate.

(b) Failure of classical energy minimization. Use the force law to obtain the total energy as a function of radius. What radius corresponds to the lowest possible energy?

(c) For the lowest-energy state, demand that the orbit circumference be one de Broglie wavelength. Obtain an expression for the total energy as a function of radius. Note how a larger radius decreases the kinetic and increases the potential energy, whereas a smaller radius increases the kinetic and decreases the potential energy.

(d) Take the derivative of the energy versus radius function and find the radius that minimizes the total energy. How large is that radius for a hydrogen atom? For a He^+ ion with one electron?

The total energy of the classical system is:

$$E = \frac{p^2}{2m} - \frac{k_e e^2}{r} \tag{31}$$

Using the force equation, we can express the kinetic energy in terms of radius, noting that the electric force must provide the centripetal force:

$$\frac{mv^2}{r} = \frac{k_e e^2}{r^2} \tag{32}$$

$$\frac{1}{2}mv^2 = \frac{k_e e^2}{2r} = \frac{p^2}{2m} \implies \qquad E = -\frac{k_e e^2}{2r} = \frac{1}{2}U \tag{33}$$

The total energy is half the potential energy, as it must be for a stable bound obit in a $1/r^2$ central force. By inspection, we can see that the minimum energy is when $r \to 0$, corresponding $E \to -\infty$, i.e., the electron crashes into the proton. Bad!

If we demand that the circumference of the orbit, $2\pi r$, equal the de Broglie wavelength,

$$2\pi r = \lambda = \frac{h}{mv} \qquad \implies \qquad mvr = \frac{h}{2\pi} = \hbar \qquad \implies \qquad mv = p = \frac{\hbar}{r} \tag{34}$$

Substituting this into the energy equation,

$$E = \frac{p^2}{2m} - \frac{k_e e^2}{r} = \frac{\hbar^2}{2mr^2} - \frac{k_e e^2}{r} \tag{35}$$

Minimizing this with respect to r,

$$\frac{dE}{dr} = \frac{-2\hbar^2}{2mr^2} + \frac{k_e e^2}{r^2} = 0 \tag{36}$$

$$\frac{\hbar^2}{mr} = k_e e^2 \tag{37}$$

$$r = \frac{\hbar^2}{e^2 m k_e} \tag{38}$$

For hydrogen, we find $r \approx 5 \times 10^{-11}$ m. For He^+, we have two protons in the nucleus (charge $2e$), and the electrical potential energy is modified to $-2k_e e^2/r$. This gives $r \approx 2.6 \times 10^{-11}$ m.

8. (a) *Early neutron model.* The neutron is an electrically neutral particle with a mass approximately equal to the proton mass. An early model considered the neutron to be an object where the electron is confined inside the proton. Assuming that the proton radius is $r \approx 10^{-15}$ m, estimate the electron's kinetic energy due to Heisenberg uncertainty, and compare it to the neutron rest mass.

(b) *Energy spread of electron beam.* A monochromatic beam of electrons of energy $E = 1\,\mathrm{keV}$ is incident on a shutter that opens for $\Delta t = 1\,\mathrm{ns}$. What is the fractional energy spread $\Delta v/v$ of the electron velocity v after the shutter? (Decide first whether to perform a relativistic or a non-relativistic calculation.)

(a) Confinement on a scale r means that the uncertainty in position is $\Delta x \approx r$, and uncertainty dictates

$$\Delta p \Delta x \approx r \Delta p \approx \hbar \qquad \Longrightarrow \qquad \Delta p \approx \frac{\hbar}{r} \tag{39}$$

The energy of the electron is $E^2 = m^2c^4 + p^2c^2$. The uncertainty in energy comes only from the second term, since the electron rest mass is fixed, and thus

$$\Delta E \approx c\Delta p \approx \frac{\hbar c}{2r} \approx 0.1\,\mathrm{GeV} \tag{40}$$

This is about 10% of the neutron's rest mass, a sizable fraction.

(b) Since the energy scale is far less than the electron's rest energy ($E \ll mc^2$), we may neglect relativistic effects. If the pulse is Δt in length, it has a spatial extent of $\Delta x = v\Delta t$. This implies an uncertainty in momentum:

$$\Delta x \Delta p = v \Delta t \Delta p \approx \frac{\hbar}{2} \qquad \Longrightarrow \qquad \Delta p = m \Delta v \approx \frac{\hbar}{2v\Delta t} \tag{41}$$

The uncertainty in velocity can be related to the uncertainty in momentum:

$$\frac{\Delta v}{v} = \frac{m\Delta v}{mv} = \frac{\hbar}{2v\Delta t}\frac{1}{mv} = \frac{\hbar}{2mv^2\Delta t} \tag{42}$$

The kinetic energy of an electron in the beam is $\frac{1}{2}mv^2$, and thus we can relate the uncertainty in velocity to the time uncertainty and the electron energy:

$$\frac{\Delta v}{v} = \frac{\hbar}{4E\Delta t} \approx 10^{-10} \tag{43}$$