University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair Spring 2010

Problem Set 6 & 7

Instructions:

1. Answer all questions below.
2. Show your work for full credit.
3. All problems are due Fri 12 March 2010 by the end of the day.
4. You may collaborate, but everyone must turn in their own work.

1. The energies of the stationary states of hydrogen slightly depend on the orbital angular momentum quantum number l. An improved formula for the energy of the state of quantum numbers n and l for nonzero l is

$$E_{n,l} = \frac{m_e e^4}{2\left(4\pi\epsilon_o\right)^2 \hbar^2 n^2}\left[1 + \frac{e^4}{\left(4\pi\epsilon_o\right)^2 \hbar^2 c^2 n}\left(\frac{1}{1+l+\frac{1}{2}\pm\frac{1}{2}} - \frac{3}{4n}\right)\right] \tag{1}$$

where the term $\pm\frac{1}{2}$ corresponds to the spin parallel and antiparallel, respectively, to the orbital angular momentum.

(a) For the case of the first excited state, $n=2$, $l=1$, and the spin antiparallel to the orbital angular momentum, find the effective difference *in electron volts* between the energy calculated according to Bohr theory and the energy calculated according to the improved formula above.
(b) For $n=2$, $l=1$, find the difference in electron volts between the energies of spin parallel and antiparallel to the angular momentum calculated according to the improved formula above. Which of these states has the lowest energy?

2. The wave function for an electron in the $2p$ state of hydrogen is given by

$$\psi_{2p} = \frac{1}{\sqrt{3}\left(2a_o\right)^{3/2}}\frac{r}{a_o}e^{-r/2a_o} \tag{2}$$

where a_o is the Bohr radius, $a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 0.529\times10^{-10}\,\text{m}$. What is the most likely distance from the nucleus to find an electron in the $2p$ state? Express your answer in terms of a_o.

3. Repeat the previous problem for an electron in the $2s$ state, where

$$\psi_{2s} = \frac{1}{4\sqrt{2\pi}}\left(\frac{1}{a_o}\right)^{3/2}\left(2-\frac{r}{a_o}\right)e^{-r/2a_o} \tag{3}$$

4. In the quantum theory of diamagnetism, one can write the susceptibility in terms of the mean square distance of electrons from the nucleus $\langle r^2 \rangle$:

$$\chi_d = \frac{N\mu}{H} = -\frac{\mu_0 Z e^2 n}{6m}\langle r^2 \rangle$$

where n is the number of atoms per unit volume, Z the atomic number, e and m are the electron charge and mass. The wavefunction of the hydrogen atom in its ground state ($1s$) is

$$\psi_0 = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

where a_o is the Bohr radius. Show that $\langle r^2 \rangle = 3a_0^2$, and calculate the diamagnetic susceptibility of atomic hydrogen. You may assume that the number of atoms per unit volume is given by Loschmidt's number, $n_0 = 2.687 \times 10^{25}\,\mathrm{m}^{-3}$ (*i.e.*, calculate χ_d at STP).

Hint: Remember that the volume element in spherical coordinates is $dV = r^2 \sin\theta \, dr \, d\theta \, d\varphi$ when you try to find $\langle r^2 \rangle$. The following integral will be useful:

$$\int_0^\infty x^n e^{-ax} = \frac{n!}{a^{n+1}}$$

5. (a) Evaluate the expectation values of the position $\langle x \rangle$ for a particle in the ground state of the one-dimensional simple harmonic oscillator, where:

$$\psi_0 = \sqrt{\frac{1}{a\sqrt{\pi}}} e^{-x^2/2a^2} \tag{4}$$

(b) Evaluate the expectation value of the position $\langle x \rangle$ for a particle in the first excited state of the one-dimensional simple harmonic oscillator. The wave function is:

$$\psi_1 = \sqrt{\frac{1}{2a\sqrt{\pi}}} \left(\frac{2x}{a}\right) e^{-x^2/2a^2} \tag{5}$$

where $a = \sqrt{\frac{\hbar}{m\omega_o}}$ and $\omega_o = \sqrt{k/m}$.

(c) Evaluate $\langle x^2 \rangle$ for the ground state of the simple harmonic oscillator in one dimension. What

is the expectation value of the potential energy of the particle?
(d) Find the uncertainty in position, $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ for the ground state.

The following integrals may be useful:

$$\int_0^\infty x^2 e^{-ax^2}\, dx = \frac{1}{4}\sqrt{\frac{\pi}{a^3}} \tag{6}$$

$$\int_{-\infty}^\infty x^3 e^{-ax^2}\, dx = \int_{-\infty}^\infty x e^{-ax^2}\, dx = 0 \tag{7}$$

$$\int_0^\infty x^4 e^{-ax^2}\, dx = \frac{3}{8}\sqrt{\frac{\pi}{a^5}} \tag{8}$$

6. Electromagnetic radiation of wavelength 20 nm is incident on atomic hydrogen. Assuming that an electron in its ground state is ionized, what is the maximum velocity at which it may be emitted?

7. An excited argon ion in a gas discharge radiates a spectral line of wavelength 450 nm. The transition from the excited to the ground state that produces this radiation takes an average time of 10^{-8} s. What is the inherent width of the spectral line $\Delta\lambda/\lambda$? *Hint: uncertainty principle.*

8. The typical operating voltage for an electron microscope is $\Delta V = 50$ kV. **(a)** What is the smallest feature one could hope to resolve? **(b)** What is the equivalent resolution if neutrons are used? **(c)** Explain in words why electrons are used, and not protons or neutrons.

9. The neutral hydrogen atom in its normal state behaves in some respects like an electric charge distribution which consists of a point charge of magnitude e surrounded by a distribution of negative charge whose density is given by

$$-\rho(r) = Ce^{-2r/a_o}$$

Here a_o is the *Bohr radius*, 0.53×10^{-10} m, and C is a constant with the value required to make the total amount of negative charge exactly e.

(a) What is the net electric charge inside a sphere of radius a_o?
(b) What is the electric field strength at this distance from the nucleus?
(c) What is C?