University of Alabama
Department of Physics and Astronomy

PH 253 / LeClair Spring 2010

Problem Set 8 Solutions

1. *Multiplicity of atomic magnetic moments.* Calculate the magnetic moments that are possible for the $n=3$ level of Hydrogen, making use of the quantization of angular momentum. You may neglect the existence of spin. Compare this with the Bohr prediction for $n=3$.

The magnetic moment can be related to the total orbital angular momentum:

$$|\vec{\mu}| = -\frac{e}{2m_e}|\vec{L}| = -\mu_B \frac{|\vec{L}|}{\hbar} \tag{1}$$

In turn, we know how the magnitude of $\vec{L}$ depends on l:

$$|\vec{L}| = \sqrt{l\,(l+1)}\hbar \tag{2}$$

So in general the moments will be

$$|\vec{\mu}| = -\mu_B \sqrt{l\,(l+1)} \tag{3}$$

For $n=3$, we may have $l=\{0,1,2\}$, so this gives $|\vec{\mu}|=\left\{0,-\sqrt{2},-\sqrt{6}\right\}\mu_B$.

In the Bohr model, our general relationship between $\vec{L}$ and $\vec{\mu}$ remains valid, but angular momentum is not given by a separate quantum number (the Bohr model has only the principle quantum number n), but simply by $L=n\hbar$. Thus, the moment for a given n in the Bohr model is single-valued, and given by

$$|\vec{\mu}_{\text{Bohr}}| = -\mu_B \frac{|\vec{L}|}{\hbar} = -n\mu_B \tag{4}$$

The Bohr model is in sharp disagreement with the full quantum solution.

2. *Transitions in a magnetic field.* Transitions occur in an atom between $l=2$ and $l=1$ states in a magnetic field of 0.6 T, obeying the selection rules $\Delta m_l=0,\pm 1$. If the wavelength before the field was turned on was 500.0 nm, determine the wavelengths that are observed. You may find the following relationship from last week's homework useful:

$$\left|\Delta\lambda\right| = \frac{\lambda^2 \Delta E}{hc} \tag{5}$$

Recall that the Zeeman effect changes the energy of a single-electron atom in a magnetic field by

$$\Delta E = m_l \left(\frac{e\hbar}{2m_e}\right) B \qquad \text{with} \qquad m_l = -l, -(l-1), \ldots, 0, \ldots, l-1, l \tag{6}$$

For convenience, note that $e\hbar/2m_e = \mu_B \approx 57.9\,\mu\text{eV/T}$, and neglect the existence of spin. See also: Pfeffer & Nir 3.2.2

In a magnetic field B, the energy levels for a given l state will split according to their value of m_l. If the original energy of the level is E_l, then the original level will be split symmetrically into $2l+1$ sub-levels, with adjacent levels shifted by $\mu_B B$:

$$E_{l,m_l} = E_l + m_l \mu_B B \tag{7}$$

This is shown schematically below for $l=2$ and $l=1$ levels. The $l=2$ level has possible m_l values of $m_l=\{-2,-1,0,1,2\}$, and thus in a magnetic field B what was a single level is now 5 individual levels. For $l=1$, we have m_l values of only $m_l=\{-1,0,1\}$, and the original level becomes a triplet upon applying a magnetic field.

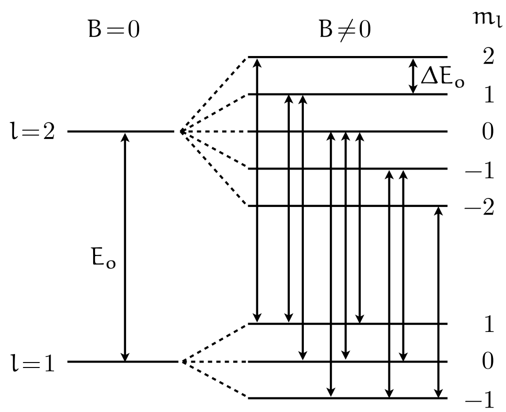


Figure 1: *Allowed transitions from $l=2$ to $l=1$ with a magnetic field applied.*

Before calculating anything, we can apply the dipole selection rules, which states that m_l can change by only $0, \pm 1$. This means that, for example, from the $l=2$, $m_l=1$ level an electron may "jump" to the any of the $l=1$, $m_l=\{2,1,0\}$ levels. On the other hand, from $l=2$, $m_l=2$ level an electron may only jump to the $l=1$, $m_l=1$ level. Following these rules, we see from the figure above that there are only 9 possible transitions allowed. Further, noting that the levels are equally spaced, we have in fact only three different transition energies.

The spacing between the levels ΔE_o is the Zeeman energy given above, $\Delta E_o = \mu_B B$. From our schematic above, it is clear that the only possible transition energies in a magnetic field are the original transition energy (no change in m_l), or the original transition energy plus or minus ΔE_o (m_l changes by ±1). The original transition energy E_o is readily found from the given wavelength $\lambda = 500$ nm:

$$E_o = \frac{hc}{\lambda} \approx 2.5\,\text{eV} \tag{8}$$

Thus, the new transition energies must be

$$E_o \longmapsto \{E_o - \Delta E_o, E_o, E_o + \Delta E_o\} = \{E_o - \mu_B B, E_o, E_o + \mu_B B\} \tag{9}$$

That is, the original transition energy plus two new ones. We can easily convert these two new energies into two new wavelengths by the energy-wavelength relationship $E = hc/\lambda$. However, this does require some numerical precision (i.e., carrying at least 7-8 digits in your calculations, and knowing the requisite constants to commensurate precision), and it is somewhat easier to simply calculate the change in energy by itself. Since we know the energy changes by $\pm\Delta E_o$, using the formula given we have

$$\left|\Delta\lambda\right| = \frac{\lambda^2 \Delta E_o}{hc} = \frac{\lambda^2 \mu_B B}{hc} \approx 0.007\,\text{nm} \tag{10}$$

The shift in energy of ΔE_o implies a shift in wavelength of $\Delta\lambda \approx 0.007$ nm, meaning the new transitions must be at the original wavelength λ plus or minus $\Delta\lambda$:

$$\lambda \longmapsto \{\lambda - \Delta\lambda, \lambda, \lambda + \Delta\lambda\} = \{499.994, 500.000, 500.007\}\ \text{nm} \tag{11}$$

3. *Stern-Gerlach experiment.* A beam of *free electrons* moves perpendicularly through a uniform magnetic field of 0.8 T. What is the energy difference between the electrons whose spins are "aligned" and "anti-aligned" with the magnetic field? See also: Pfeffer & Nir 3.4.1-2

The energy difference between the spins parallel and antiparallel to the magnetic field is due to the spin magnetic moments interacting with the magnetic field. For a moment $\vec{\mu}$ in a field B, the energy is

$$E = -\vec{\mu} \cdot \vec{B} \tag{12}$$

For a free electron, we have no orbital angular momentum. The magnetic moment is only due to the electrons' spin.[i] Measured along a particular axis, the spin magnetic moment can *only* take on two values, $\pm\mu_B$ – that is, a moment of magnitude μ_B directed either parallel or antiparallel to the magnetic field. For an electron spin parallel to the magnetic field, the energy is then $E_{\uparrow\uparrow} = -\mu_B B$, while for antiparallel spins the energy is $E_{\uparrow\uparrow} = +\mu_B B$. Their difference is then

$$\Delta E = 2\mu_B B \approx 9.3 \times 10^{-5}\,\mathrm{eV} \tag{13}$$

4. *Radio astronomy.* The hydrogen $\lambda = 21\,\mathrm{cm}$ line is used in radio astronomy to map the galaxy. The line arises from the emission of a photon when the electron in a galactic hydrogen atom "flips" its spin from being aligned to being anti-aligned with the spin of the proton in the hydrogen atom.[ii] What is the magnetic field the electron experiences to induce this spin flip?

If the transition is due to a spin flip in the presence of a magnetic field B, then based on the results of the previous problem we must expect that the electron gains or loses an energy $\Delta E = 2\mu_B B$. This difference in energy must correspond to the $\lambda = 21\,\mathrm{cm}$ emission, and thus

$$\Delta E = 2\mu_B B = \frac{hc}{\lambda} \qquad \Longrightarrow \qquad B = \frac{hc}{\mu_B \lambda} \approx 0.051\,\mathrm{T} \tag{14}$$

[i] Remember, owing to uncertainty we can only know a single component of the spin moment. If we choose the field to be applied along the z axis, then the application of a magnetic field projects out the z component of the spin moment, which can only take on values $\mu_{s,z} = \pm\mu_B$.

[ii] Even though this process is strongly forbidden, the abundance of hydrogen in the galaxy is sufficiently enormous to make observation practical.

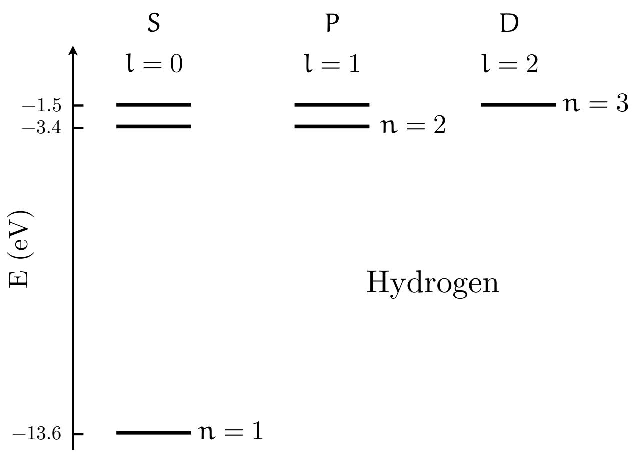
5. *Dipole selection rules.*

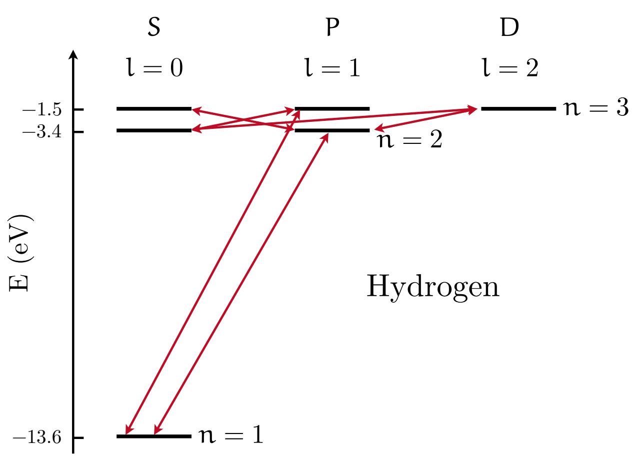
(a) For hydrogen, the energy levels through $n=3$ are shown below. What are the possible electric dipole transitions for these states? It may be convenient to simply draw arrows in the diagram. Recall the "selection rules" for electric dipole transitions, $\Delta l = \pm 1$. Spin may be ignored.

(b) Repeat for para- and ortho-helium, also shown below, treating both as distinct atoms.[iii]

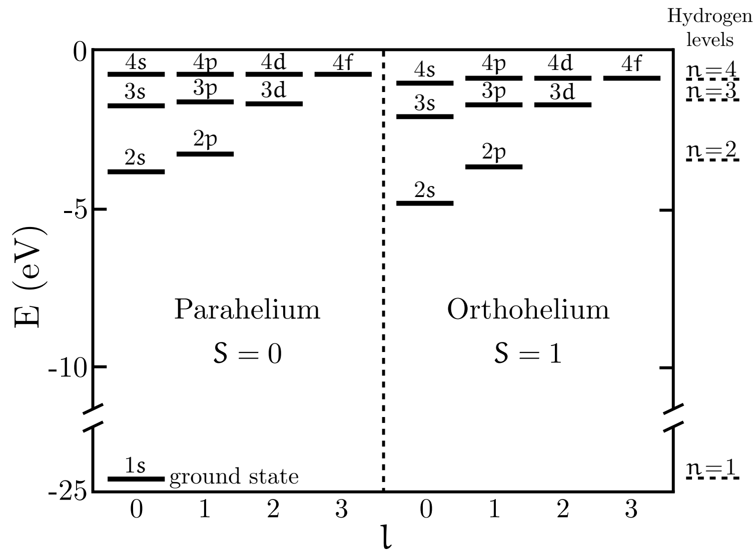
See also: Pfeffer & Nir 3.3.2

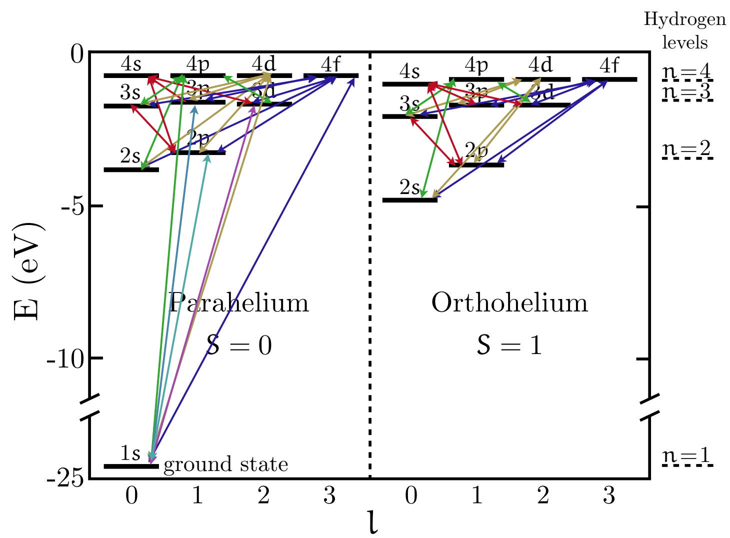
You can change as many levels at a time as you like (i.e., move up or down arbitrarily), but the $\Delta l = \pm 1$ selection rule means you can must move one (and only one) unit to the left or right to make a transition. Thus, the problem is reduced to drawing lines in the figures given. Note that with the energy scales given, not all of the transitions are in the visible range – the longest lines will be in the ultraviolet, the shortest in the infrared (or microwave, for closely-spaced He levels). For the very closely-spaced levels (e.g., $3p$ and $3d$) the transitions for He have been omitted for clarity; the diagram quickly becomes quite complicated!





[iii]Two types of helium: para-helium, with the two electron spins parallel ($S=0$), and ortho-helium, with the two electron spins antiparallel ($S=1$). According to the dipole selection rules, helium atoms cannot change by a radiative





process from one to the other, as this would not conserve angular momentum, so ortho- and para-helium behave largely as distinct atoms. (Forbidden transitions are not strictly forbidden, but violating the selection rules incurs a cost of ~10^5 in transition probability).

As the energy level diagram shows, the lowest state corresponds to para-helium, and the next highest excited state ortho-helium. The ortho-helium excited state can be reached by electrical discharge excitation, a non-radiative process (i.e., not obeying the same selection rules.) This excited state is very long lived (~10 ms) because returning to the ground state would violate selection rules.

6. *Splitting of Sodium D lines.* The electron's intrinsic magnetic moment $\vec{\mu}_s$ and intrinsic spin angular momentum $\vec{S}$ are proportional to each other; their relationship can be written as

$$\vec{\mu}_s = -g_s \frac{e}{2m}\vec{S} = -g\mu_b\vec{S} \tag{15}$$

with $g_s \approx 2$. The energy of the electron in a effective magnetic field $\vec{B}$ is $E = -\vec{\mu}_s \cdot \vec{B}$.

In Sodium, transitions occur between two spin-orbit-split $L=1$ states and a single $L=0$ state, leading to emission lines at 588.995 nm and 589.592 nm. Estimate the strength of the effective magnetic field produced by the electron's orbital motion (i.e., the effective field due to the spin-orbit interaction) which results in this wavelength difference. You may wish to make use of the relationship given in problem 2. See also: Pfeffer & Nir 3.4.3,

`http://hyperphysics.phy-astr.gsu.edu/HBASE/quantum/sodium.html#c2`

The transition occurs between an $L=1$ state and an $L=0$ state; only the $L=1$ state will split due to the spin-orbit interaction. The difference in energy between the $L=1$, $m_s = \frac{1}{2}$ and $L=1$, $m_s = -\frac{1}{2}$ states will be $\Delta E = 2\mu_B B$ in an effective magnetic field B. The energy difference can also be equated to the wavelength difference, using the relationship given in problem 2:

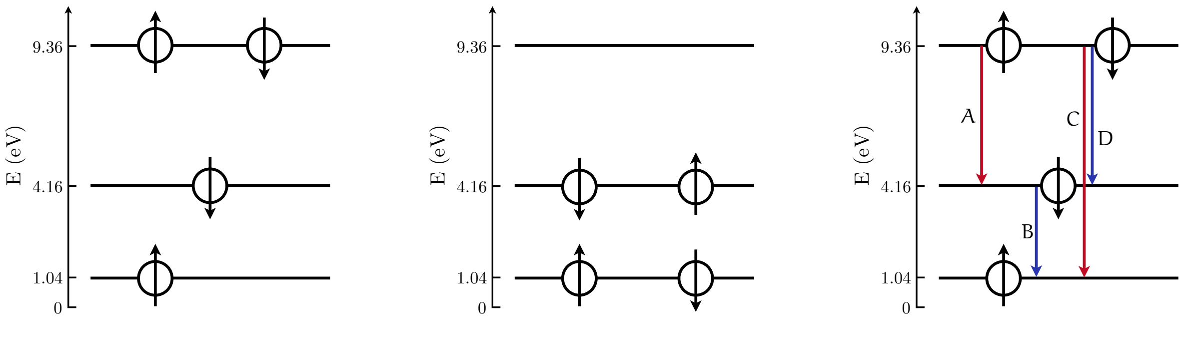
$$\Delta E = E(m_s = \frac{1}{2}) - E(m_s = -\frac{1}{2}) \approx \frac{hc|\Delta\lambda|}{\lambda^2} = 2\mu_B B \tag{16}$$

Solving for B, with $\Delta\lambda = 589.592 - 588.995 \approx 0.597$ nm and $\lambda = (589.592 + 588.995)/2 \approx 589.294$ nm,

$$B = \frac{hc\Delta\lambda}{2\mu_B\lambda^2} = \frac{(1240\,\mathrm{eV\cdot nm})\,(0.597\,\mathrm{nm})}{2\,(57.9\times10^{-6}\,\mathrm{eV/T})\,(589.592\,\mathrm{nm})^2} \approx 18.4\,\mathrm{T} \tag{17}$$

7. *Pauli exclusion.* What are the energies of the photons that would be emitted when the four-electron system in the figure below returns to its ground state? See also: Pfeffer & Nir 3.4.5

The ground state must consist of a spin up and spin down electron in each of the two lowest energy levels. The possible transitions returning the system to the ground state are shown above. Transitions A and D have the same energy difference, and thus give rise to the same energy photon, and thus only three different photons should be observed:



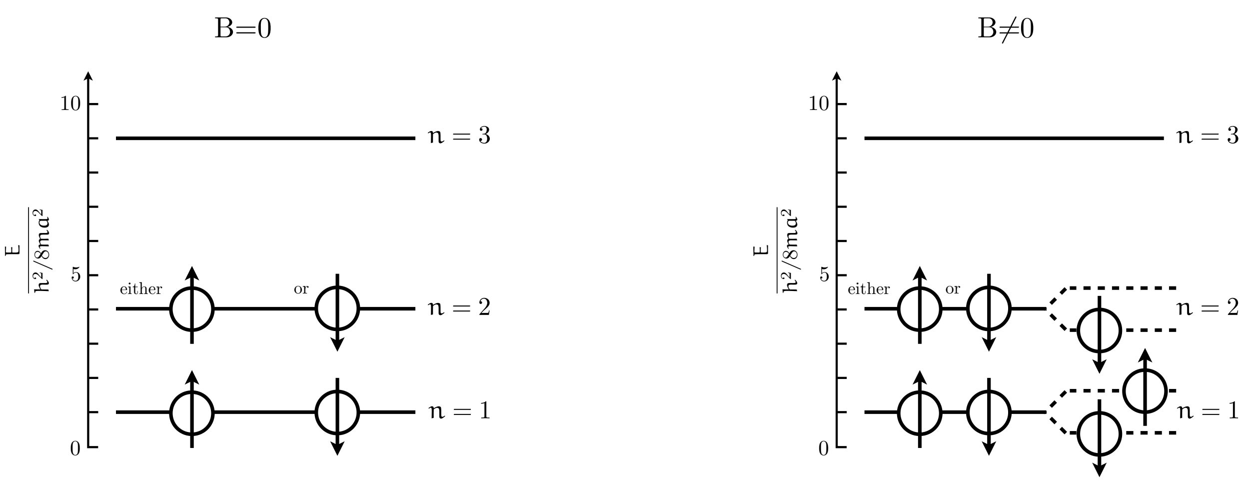
A system of four electrons with three energy levels, their ground state, and the possible transitions to the ground state.

$$E_A = E_D = E_3 - E_2 = 9.36\,\mathrm{eV} - 4.16\,\mathrm{eV} = 5.20\,\mathrm{eV} \tag{18}$$

$$E_B = E_2 - E_1 = 4.16\,\mathrm{eV} - 1.04\,\mathrm{eV} = 3.12\,\mathrm{eV} \tag{19}$$

$$E_C = E_3 - E_1 = 9.36\,\mathrm{eV} - 1.04\,\mathrm{eV} = 8.32\,\mathrm{eV} \tag{20}$$

8. Three non-interacting particles are in their ground state in an infinite square well;[iv] see the figure above. What happens when a magnetic field is turned on which interacts with the spins of the particles? Draw the new levels and particles (with spin).



A system of three electrons in an infinite square well, and the shift of the energy levels when a magnetic field is applied.

After the external magnetic field is applied, the new value of each particle's energy level E_i equals

[iv]Recall the energies in an infinite square well are $E = n^2h^2/8ma^2$

the original level E_n plus the interaction (Zeeman) energy:

$$E_i = E_n - \vec{\mu}_s \cdot \vec{B} = E_n - \mu_{sz} B = E_n + 2m_s \mu_B B \tag{21}$$

Since $m_s = \pm\frac{1}{2}$, the new levels will be displaced from the old by an amount $\Delta E = \pm\mu_B B$, with a spin $-\frac{1}{2}$ ("down") particle occupying the lower sublevel and a spin $+\frac{1}{2}$ ("up") particle occupying the upper sublevel. In contrast to the situation without a magnetic field present, the particle in the $n = 2$ level will have its spin $-\frac{1}{2}$ if the system is in the ground state.