Problem Set 1: Solutions

1. An atom of mass $m_1 = m$ moves in the positive x direction with velocity $v_1 = v$. It collides with and sticks to an atom of mass $m_2 = 2m$ moving in the positive y direction with speed $v_2 = \frac{2}{3}v$. **(a)** Find the resultant speed and direction of motion of the combination. **(b)** Find the kinetic energy lost in this inelastic collision.

Solution: **(a)** Since this is an inelastic collision, all we can really do is conserve momentum. Initially, m_1 moves along the x axis and m_2 along the y axis, so the initial components of momentum are:

$$p_{xi} = m_1 v_1 \tag{1}$$
$$p_{yi} = m_2 v_2 \tag{2}$$

After the collision, we have a single object of mass $m_1 + m_2$, and it will move with some angle θ with respect to the x axis at speed v_f. Its components of momentum are:

$$p_{xf} = (m_1 + m_2)\, v_f \cos\theta \tag{3}$$
$$p_{yf} = (m_1 + m_2)\, v_f \sin\theta \tag{4}$$

Conservation of momentum gives:

$$p_{xi} = p_{xf} \tag{5}$$
$$m_1 v_1 = (m_1 + m_2)\, v_f \cos\theta \tag{6}$$
$$p_{yi} = p_{yf} \tag{7}$$
$$m_2 v_2 = (m_1 + m_2)\, v_f \sin\theta \tag{8}$$

The easiest thing to do now is to divide Eq. 8 by Eq. 6, and note $m_2 = 2m$, $v_1 = v$, and $v_2 = 2/3v$:

$$\frac{m_2 v_2}{m_1 v_1} = \tan\theta = \frac{2m \cdot \frac{2}{3}v}{mv} = \frac{4}{3} \tag{9}$$
$$\implies \qquad \theta = \tan^{1}\left(\frac{4}{3}\right) \approx 53.1^\circ \approx 0.927\,\text{rad} \tag{10}$$

That gives the direction of the combined mass, the speed is then easily found from either Eq. 6 or Eq. 8. Note that if $\tan\theta = 4/3$, we have a 3-4-5 triangle, so $\sin\theta = 4/5$ and $\cos\theta = 3/5$.

$$v_f = \frac{m_2 v_2}{(m_1 + m_2)\sin\theta} = \frac{2m \cdot \frac{2}{3}v}{(m + 2m)\left(\frac{4}{5}\right)} = \frac{5}{9}v \tag{11}$$

(b) Finding the kinetic energy lost is just a matter of calculating it before and after the collision. Initially, masses m_1 and m_2 travel at velocities v_1 and v_2:

$$K_i = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}mv^2 + \frac{1}{2}(2m)\left(\frac{2}{3}v\right)^2 = \frac{17}{18}mv^2 \tag{12}$$

After the collision, the combined mass travels at v_f:

$$K_f = \frac{1}{2}(m_1 + m_2)v_f^2 = \frac{1}{2}(m + 2m)\left(\frac{5}{9}\right)^2 v^2 = \frac{25}{54}mv^2 \tag{13}$$

The loss in kinetic energy is then

$$\Delta K = K_i - K_f = \frac{17}{18}mv^2 - \frac{25}{54}mv^2 = \frac{13}{27}mv^2 \tag{14}$$

$$\frac{\Delta K}{K_i} = \frac{26}{51} \approx 51\% \tag{15}$$

2. **(a)** On the unrealistic assumption that there are no other charged particles in the vicinity, at what distance below a proton would the upward force on an electron equal its weight? **(b)** What is the induced EMF between the ends of the wingtips of a Boeing 737 when it is flying over the magnetic north pole? Google has the numbers you require.

Solution: **(a)** This just means that the attractive force between the proton (charge e) and electron (charge $-e$) at a distance d must balance the weight of the electron:

$$\frac{ke^2}{d^2} = mg \qquad \Longrightarrow \qquad d = \sqrt{\frac{ke^2}{mg}} \approx 5.08\,[\mathrm{m}] \tag{16}$$

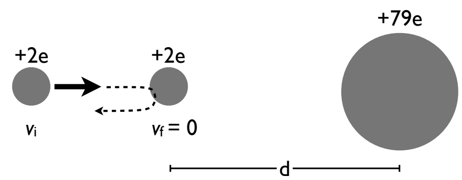
(b) The induced voltage can be found by considering the motion of the conducting metal plane in a perpendicular magnetic field, and making a few rough but justifiable assumptions.

First, at the south magnetic pole, the magnetic field will be essentially straight down. If the 737 is flying level over the ground, this means that its metal (conducting) skin is in motion relative to a magnetic field. This in turn means that there will be a motionally-induced voltage. If the field is straight down, and the 737 travels straight forward, then positive charges will experience a force in the port (left) direction, and negative charges toward the starboard (right). This means that the wingtips will have a potential difference between them due to the magnetic force on the charges in the conducting skin. If the wingspan is l meters, the airplane's velocity v and the vertical magnetic field B, then we know the potential difference due to motion in a magnetic field is $\Delta V = Blv$.

The wingspan of a 737 is roughly 30 m, and its cruising speed is about 200 m/s.[i] Currently, the earth's magnetic field[ii] at the south magnetic pole[iii] is about 60 μT. Putting this together,

$$\Delta V = Blv = (60\,\mu\text{T})\,(30\,\text{m})\,(200\,\text{m/s}) \approx 0.36\,\text{V}$$

3. In Rutherford's famous scattering experiments that led to the planetary model of the atom, alpha particles (having charge $+2e$ and masses of 6.64×10^{-27} kg) were fired toward a gold nucleus with charge $+79e$. An alpha particle, initially very far from the gold nucleus, is fired at a speed of $v_i = 2.00 \times 10^7$ m/s directly toward the nucleus, as shown below. How close does the alpha particle get to the gold nucleus before turning around? Assume the gold nucleus remains stationary, and that energy is conserved.



Solution: We can treat this as a conservation of energy problem. The two energies of interest are the kinetic energy of the alpha particle, and the potential energy of the alpha particle-gold nucleus pair. Since the two are both positively charged, they will repel each other, and their electrical potential energy will be positive at any finite distance. If the alpha particle is initially very, very far away, we can approximate their starting separation as infinite, meaning their initial electrical potential energy is zero, and the total energy of the system is just the alpha particle's kinetic energy.

How close does the alpha particle get? When it used all of its initial kinetic energy up as electrical potential energy. The closer it gets, the larger the electrical potential energy, and the more kinetic energy it must spend. At some point, it is all gone, and the particle instantaneously stops and then turns around. At that point of closest approach, the alpha particle's kinetic energy is zero. Comparing the energy in the initial and final cases will allow us to find the distance of closest approach d.

To go further, we must make one approximation: even though $v/c \sim 0.067$, and we should in principle use the relativistic equation for kinetic energy, we will ignore it for simplicity. The error we will incur will be only of order v/c, so $\lesssim 10\%$ at best. With that out of the way, we can apply conservation of energy. Let the alpha particle have charge q_a and the gold nucleus charge q_g, we can plug in numbers later.

[i] http://en.wikipedia.org/wiki/Boeing_737
[ii] http://www.ngdc.noaa.gov/geomag/magfield.shtml
[iii] http://en.wikipedia.org/wiki/South_Magnetic_Pole

$$K_i + U_i = K_f + U_f \tag{17}$$

$$\frac{1}{2}mv_i^2 + 0 = 0 + \frac{k_e q_a q_g}{d} \tag{18}$$

$$d = \frac{2k_e q_a q_g}{mv_i^2} \approx \frac{\left(8.99 \times 10^9\right)\left(2 \cdot 1.60 \times 10^{-19}\right)\left(79 \cdot 1.60 \times 10^{-19}\right)}{\left(6.64 \times 10^{-27}\right)\left(2.00 \times 10^7\right)^2} \tag{19}$$

$$\approx 2.74 \times 10^{-14}\,\mathrm{m} = 27.4\,\mathrm{fm} \tag{20}$$

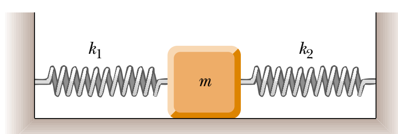
What is the acceleration at the moment it reverses? At this point the alpha particle is a distance d from the gold nucleus. The force between them at that instant is just the electric force, and the alpha particle's acceleration will be its net force divided by its mass. Basically: acceleration comes from force, and there is only one force present here.

$$a = \frac{1}{m}F = \frac{1}{m}\frac{k_e q_a q_g}{d^2} \tag{21}$$

Using our previous expression for d,

$$a = \frac{k_e q_a q_g}{md^2} = \frac{k_e q_a q_g}{m}\frac{m^2 v_i^4}{4k_e^2 q_a^2 q_g^2} = \frac{mv_i^4}{4k_e q_a q_g} \approx 7.3 \times 10^{27}\,\mathrm{m/s^2} \tag{22}$$

4. A block of mass m is connected to two springs of force constants k_1 and k_2 as shown below. The block moves on a frictionless table after it is displaced from equilibrium and released. Determine the period of simple harmonic motion.



Solution: Say we displace the block to the right by an amount x. Both springs will try to bring the block back toward equilibrium - one will pull, one will push, but *both will act in the same direction*. That means the net force is

$$F_{\mathrm{net}} = -k_1 x - k_2 x = -\left(k_1 + k_2\right)x = ma \tag{23}$$

This is exactly the same as what we would find for a single spring, except the spring constant has become $k_1 + k_2$ rather than just k. The solution must be

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_1 + k_2}} \tag{24}$$

5. Calculate the fraction of molecules in a gas that are moving with translational kinetic energies between $0.02\,k_B T$ and $0.04\,k_B T$.

Solution: Referring to chapter 1 in your text, we calculate the number of particles in an energy range $[E_1, E_2]$ by integrating the Maxwell-Boltzmann distribution over that interval:

$$N(E_1 : E_2) = \int_{E_1}^{E_2} \frac{2N}{\sqrt{\pi}} \frac{1}{(k_B T)^{3/2}} E^{1/2} e^{-E/k_B T}\, dE \tag{25}$$

Of course now we have a problem, how to get rid of the T's? If you look carefully, you'll notice that neither energy nor temperature is really the crucial variable, it is the dimensionless quantity $E/k_B T$. This suggests a substitution $u = E/k_B T$, $du = dE/k_B T$, making the limits of integration $E_1/k_B T = 0.02$ and $E_2/k_B T = 0.04$.

$$N(u_1 : u_2) = \frac{2N}{\sqrt{\pi}} \int_{0.02}^{0.04} \sqrt{u} e^{-u}\, du \tag{26}$$

The fraction of particles in the energy range of interest is then:

$$\frac{N(u_1 : u_2)}{N} = \frac{2}{\sqrt{\pi}} \int_{0.02}^{0.04} \sqrt{u} e^{-u}\, du \approx 0.00377 \tag{27}$$

The integral is readily found numerically, e.g., using `www.wolframalpha.com`.