Problem Set 2: Solutions

1. One of the strongest emission lines observed from distant galaxies comes from hydrogen and has a wavelength of 122 nm (in the ultraviolet region). **(a)** How fast must a galaxy be moving away from us in order for that line to be observed in the visible region at 366 nm? **(b)** What would be the wavelength of the line if that galaxy were moving toward us at the same speed?

Solution: **(b)** The relativistic doppler effect relates the frequency of radiation seen by an observer in motion relative to its source. If the source and observer are moving apart, the observer sees "red shifted" radiation with a larger wavelength and smaller frequency. If the source and observer are moving toward one another, the observer sees "blue shifted" radiation with a smaller wavelength and larger frequency.

For source and observer separating, we have

$$f' = f\sqrt{\frac{1-v/c}{1+v/c}} \tag{1}$$

where f' is the frequency the observer sees. Noting $f\lambda = c$, we can write this in terms of wavelengths:

$$\lambda = \lambda'\sqrt{\frac{1-v/c}{1+v/c}} \tag{2}$$

We are given the source wavelength $\lambda = 122\,\text{nm}$ and the observer's wavelength $\lambda' = 366\,\text{nm}$. It is a matter of algebra to solve the equation above for v/c:

$$\frac{v}{c} = \frac{1-(\lambda/\lambda')^2}{1+(\lambda/\lambda')^2} \approx 0.8 \tag{3}$$

(b) For source and observer moving toward one another, we simply flip the plus and minus signs in our previous equations:

$$f' = f\sqrt{\frac{1+v/c}{1-v/c}} \tag{4}$$

$$\lambda = \lambda'\sqrt{\frac{1+v/c}{1-v/c}} \tag{5}$$

This also amounts to swapping primed and unprimed variables. In any event, given $\lambda = 122\,\text{nm}$ and

$v = 0.8c$, we find

$$\lambda' = \lambda\sqrt{\frac{1 - v/c}{1 + v/c}} \approx 40.7\,\text{nm} \tag{6}$$

2. Two spaceships approach the Earth from opposite directions. According to an observer on the Earth, ship A is moving at a speed of $0.753c$ and ship B at a speed of $0.851c$. What is the velocity of ship A observed from ship B? Of ship B observed from ship A?

Solution: Remember, for velocity addition all you need to do is figure out whether you should add or subtract the velocities according to your everyday intuition, then use the corresponding relativistic formula. In this case, with the ships moving head on, we'd clearly want to add their velocities together, and it wouldn't matter if we were in ship A or ship B. The same conclusion holds from either point of view. That leaves us with just one possible relativistic formula. We'll use subscripts to denote which ship's speed is begin measured, and superscripts to denote the frame of reference:

$$v_b^a = \frac{v_a^e + v_b^e}{1 + v_a^e v_b^e/c^2} = \frac{0.753c + 0.851c}{1 + (0.753c)\,(0.851c)\,/c^2} \approx 0.978c \tag{7}$$

We would also have to conclude that $v_a^b = v_b^a$, since both of these quantities just represent the relative velocity of the two ships as measured by them.

3. Derive the Lorentz velocity transformations for v'_x and v'_z.

Solution: We can start with v'_x, suspecting that if we figure it out, v'_z ought to be simple. The velocity v'_x is just dx'/dt', so from our knowledge of how x' and t' transform we should be able to find the requisite transformation. Start with the chain rule:

$$v'_x = \frac{dx'}{dt'} = \frac{dx'/dt}{dt'/dt} \tag{8}$$

The numerator and denominator are easily found from the Lorentz transformations. Note that v is the relative velocity between primed and unprimed reference frames, and as such is *constant.*

$$\frac{dx'}{dt} = \frac{d}{dt}\left(\frac{x - vt}{\sqrt{1 - v^2/c^2}}\right) \tag{9}$$

$$\frac{dt'}{dt} = \frac{d}{dt}\left(\frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}\right) \tag{10}$$

We must pause to note that $dx/dt = v_x$ is just the velocity in the unprimed frame, and $dt/dt = 1$

(meaning the entire denominator is just a constant in both cases). So long as the differentials are all in the same reference frame, it is just normal math. Proceeding with that in mind:

$$\frac{dx'}{dt} = \frac{v_x - v}{\sqrt{1 - v^2/c^2}} \tag{11}$$

$$\frac{dt'}{dt} = \frac{1 - v v_x/c^2}{\sqrt{1 - v^2/c^2}} \tag{12}$$

Applying the chain rule, we have v'_x:

$$v'_x = \frac{dx'/dt}{dt'/dt} = \frac{v_x - v}{1 - v v_x/c^2} \tag{13}$$

As for v'_z, it is easy as promised once we remember $z' = z$:

$$v'_z = \frac{dz'}{dt'} = \frac{dz'/dt}{dt'/dt} = \frac{dz/dt}{dt'/dt} = \frac{v_z}{dt'/dt} = \frac{v_z\sqrt{1 - v^2/c^2}}{1 - v v_x/c^2} \tag{14}$$

4. According to observer O, two events occur separated by a time interval of $\Delta t = +0.465\,\mu s$ at at locations separated by $\Delta x = +53.4\,m$. **(a)** According to observer O', who is in motion relative to O at a speed of $0.762c$ in the positive x direction, what is the time interval between the two events? **(b)** What is the spatial separation between the two events according to O'?.

Solution: By now you have probably realized that this is a textbook problem, and you can find answers to the odd problems in the back of the textbook. Doesn't help with showing your work, but it is a good sanity check.

We really only need the Lorentz transformations:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{15}$$

$$x' = \gamma\,(x - vt) \tag{16}$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{17}$$

Here v is the relative velocity between the two reference frames, and γ is defined for convenience. In this problem we are talking about the spatial and temporal separation of events. In the unprimed frame, we might decide the events take place at (t_1, x_1) and (t_2, x_2), giving a spatial separation of $\Delta x = x_2 - x_1$ and a temporal separation of $\Delta t = t_2 - t_1$. What we are after is how the Δx and Δt transform. Of course they transform just like individual positions and times do, but we can prove it to be certain. We just transform the individual times and positions and subtract them.

$$\Delta x' = x_2' - x_1' = \gamma(x_2 - vt_2) - \gamma(x_1 - vt_1) = \gamma(x_2 - x_1 - vt_2 + vt_1) = \gamma(\Delta x - v\Delta t) \tag{18}$$

$$\Delta t' = t_2' - t_1' = \gamma\left(t_2 - \frac{vx_2}{c^2}\right) - \gamma\left(t_1 - \frac{vx_1}{c^2}\right) = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \tag{19}$$

Of course, it had to come out this way, since the Lorentz transformations are linear in position and time. All we do now is plug in the quantities we were given: $\Delta t = +0.465\,\mu\text{s}$, $\Delta x = +53.4\,\text{m}$, and $v = 0.762c$ from which $\gamma \approx 1.544$.

$$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) \approx 0.51\,\mu\text{s} \tag{20}$$

$$\Delta x' = \gamma(\Delta x - v\Delta t) \approx -81.7\,\text{m} \tag{21}$$

The negative sign is nothing to be worried about here, it just means the second event is closer in one frame and the first is closer in the other.

5. The work-energy theorem relates the change in kinetic energy of a particle to the work done on it by an external force: $\Delta K = W = \int F\,dx$. Writing Newton's second law as $F = dp/dt$, show that $W = \int v\,dp$ and integrate by parts to obtain the result

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

Solution: This is mainly just math. Here's one way to go about it.

We are concerned with finding the kinetic energy of a particle accelerated from rest, starting with zero kinetic energy, so we may simply write $K = W = \int F\,dx$ and drop the Δ. Using what we have, and noting $v = dx/dt$:

$$K = \int_{v=0}^{v=v} F\,dx = \int_0^v \frac{dp}{dt}\,dx = \int_0^v dp\,\frac{dx}{dt} = \int_0^v v\,dp \tag{22}$$

It is this result we can integrate by parts. Let $f = dp$ and $g' = dx/dt$, and recall integration by parts gives $\int fg' = fg - \int f'g$.

$$K = \int_0^v v\,dp = pv\Big|_0^v - \int_0^v p\,dv = pv - \int_0^v p\,dv \tag{23}$$

Noting $p = mv/\sqrt{1 - v^2/c^2}$,

$$K = \frac{mv^2}{\sqrt{1-v^2/c^2}} - \int\limits_0^v \frac{mv}{\sqrt{1-v^2/c^2}}\,dv = \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2}\Big|_0^v \tag{24}$$

$$= \frac{mv^2}{\sqrt{1-v^2/c^2}} + mc^2\sqrt{1-v^2/c^2} - mc^2 = \frac{mv^2 + mc^2\left(1-v^2/c^2\right)}{\sqrt{1-v^2/c^2}} - mc^2 \tag{25}$$

$$= \frac{mc^2}{\sqrt{1-v^2/c^2}} - mc^2 = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right) mc^2 = (\gamma - 1)\, mc^2 \tag{26}$$

6. Electrons are accelerated to high speeds by a two-stage machine. The first stage accelerates the electrons from rest to $v = 0.99c$. The second stage accelerates the electrons from $0.99c$ to $0.999c$. **(a)** How much energy does the first stage add to the electrons? **(b)** How much energy does the second stage add in increasing the velocity by only 0.9%?

Solution: Don't over think this one - we just need to find the energy at each stage and subtract. It is most convenient to leave the energy in terms of electron volts, noting that the rest energy of an electron is $m_e c^2 \approx 511\,\text{keV}$. Clearly only the kinetic energy changes, since the rest energy is the same at every stage. Since we know $K = (\gamma - 1)\, mc^2$, by stage the energy is:

$$v_0 = 0 \qquad K_0 = 0 \tag{27}$$

$$v_1 = 0.99c \qquad K_1 = (7.089 - 1)mc^2 = 3.11\,\text{MeV} = 4.98 \times 10^{-13}\,\text{J} \tag{28}$$

$$v_2 = 0.999c \qquad K_2 = (22.37 - 1)mc^2 = 10.9\,\text{MeV} = 1.74 \times 10^{-12}\,\text{J} \tag{29}$$

This means that the first stage adds $K_1 - K_0 = 3.11\,\text{MeV}$ (about $5.8 \times 10^{-13}\,\text{J}$), while the second adds nearly twice as much on top of that, $K_2 - K_1 = 7.79\,\text{MeV}$.

7. A charge q at $x = 0$ accelerates from rest in a uniform electric field $\vec{\mathbf{E}}$ which is directed along the positive x axis.

(a) Show that the acceleration of the charge is given by

$$a = \frac{qE}{m}\left(1 - \frac{v^2}{c^2}\right)^{3/2}$$

(b) Show that the velocity of the charge at any time t is given by

$$v = \frac{qEt/m}{\sqrt{1 + (qEt/mc)^2}}$$

(c) Find the distance the charge moves in a time t. *Hint:* `http://integrals.wolfram.com`

Solution: We are to find the acceleration, velocity, and position as a function of time for a particle in a uniform electric field. We are given the electric force and the boundary conditions $x=0$, $v=0$ at $t=0$. We will need only $F=dp/dt$, $p=\gamma mv$, the definition of γ (given in previous problems), and a good knowledge of calculus (including the chain rule once again).

First, we must relate force and acceleration relativistically. Since velocity is explicitly a function of time here, so is γ, and we must take care.

$$F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = \gamma m \frac{dv}{dt} + mv\frac{d\gamma}{dt} \tag{30}$$

$$\frac{d\gamma}{dt} = \frac{d}{dt}\frac{1}{\sqrt{1-v^2c^2}} = \frac{\left(-\frac{1}{2}\right)\left(-\frac{2v}{c^2}\right)\frac{dv}{dt}}{(1-v^2/c^2)^{3/2}} = \frac{v}{c^2}\frac{1}{(1-v^2/c^2)^{3/2}} \tag{31}$$

$$F = \gamma m \frac{dv}{dt} + \frac{mv^2}{c^2}\frac{1}{(1-v^2/c^2)^{3/2}}\frac{dv}{dt} = m\left(\frac{dv}{dt}\right)\left(\frac{1}{\sqrt{1-v^2/c^2}} + \frac{\frac{v^2}{c^2}}{(1-v^2/c^2)^{3/2}}\right) \tag{32}$$

$$F = ma\left(\frac{1-\frac{v^2}{c^2}}{(1-v^2/c^2)^{3/2}} + \frac{\frac{v^2}{c^2}}{(1-v^2/c^2)^{3/2}}\right) = \frac{ma}{(1-v^2/c^2)^{3/2}} \tag{33}$$

If you decided not to use γ and wrote everything explicitly in terms of $1/\sqrt{1-v^2/c^2}$, that is fine. The end result is the same.

That accomplished, we can set the net force equal to the electric force qE and solve for acceleration:

$$F = qE = \frac{ma}{(1-v^2/c^2)^{3/2}} \tag{34}$$

$$a = \frac{qE}{m}\left(1-v^2/c^2\right)^{3/2} \tag{35}$$

We can find velocity by writing a as dv/dt (as it was above) and noticing that the resulting equation is separable.

$$a = \frac{dv}{dt} = \frac{qE}{m}\left(1-v^2/c^2\right)^{3/2} \tag{36}$$

$$\frac{qE}{m}\,dt = \frac{dv}{(1-v^2/c^2)^{3/2}} \tag{37}$$

We can now integrate both sides, noting from the boundary conditions that if time runs from 0 to t, the velocity runs from 0 to v.

$$\int_0^v \frac{dv}{(1-v^2/c^2)^{3/2}} = \int_0^t \frac{qE}{m}\,dt \tag{38}$$

$$\left.\frac{v}{\sqrt{1-v^2/c^2}}\right|_0^v = \left.\frac{qEt}{m}\right|_0^t \tag{39}$$

$$\frac{v}{\sqrt{1-v^2/c^2}} = \frac{qEt}{m} \tag{40}$$

Solving for v, we first square both sides ...

$$\frac{v^2}{1-v^2/c^2} = \frac{q^2E^2t^2}{m^2} \tag{41}$$

$$v^2 = \left(1-\frac{v^2}{c^2}\right)\frac{q^2E^2t^2}{m^2} \tag{42}$$

$$v^2\left(1+\frac{q^2E^2t^2}{m^2c^2}\right) = \frac{q^2E^2t^2}{m^2} \tag{43}$$

$$v = \frac{qEt/m}{\sqrt{1+(qEt/mc)^2}} \tag{44}$$

We can find position by integrating v through time from 0 to t, which is straightforward.

$$x = \int_0^t \frac{qEt/m}{\sqrt{1+(qEt/mc)^2}}\,dt = \left.\frac{mc^2}{qE}\sqrt{1+\left(\frac{qEt}{mc}\right)^2}\right|_0^t = \frac{mc^2}{qE}\left(\sqrt{1+\left(\frac{qEt}{mc}\right)^2}-1\right) \tag{45}$$

Classically, we would expect a parabolic path, but in relativity we find the path is a *hyperbola*.[i] Also note that the position, velocity, and acceleration depend overall on the ratio between the particle's rest energy mc^2 to the electric force qE (note energy/force is distance).

Double check: If we let the ratio v/c tend to zero, we should recover the classical result for acceleration:

$$a = \frac{qE}{m}\frac{1}{(1-v^2/c^2)^{3/2}} \qquad \lim_{\gamma\to 0} a = \frac{qE}{m} \tag{46}$$

As a separate check on the velocity expression, we note that if the classical acceleration is qE/m,

[i]If you square both sides, the equation for $x(t)$ can be put in the form $x^2/a^2 - t^2/b^2 = 1$, the standard form of the equation for a hyperbola.

and $v = at$ in the classical limit. Plugging that in our expression for velocity,

$$v = \frac{qEt/m}{\sqrt{1+(qEt/mc)^2}} = \frac{at}{\sqrt{1+(at/c)^2}} \tag{47}$$

If $v \ll c$, then the denominator tends toward 1, and we recover $v = at$. We can do the same for the expression for x to show that it reduces to the correct classical limit. Finally, dimensional analysis will show that the units work out correctly for all three expressions - the ratio qE/mc has units of force/momentum which gives inverse seconds, so qEt/mc is dimensionless. The ratio qE/m is force/mass which has units of acceleration, so qEt/m has units of velocity. The ratio mc^2/qE has units of energy/force which gives distance.

8. An astronaut takes a trip to Sirius, which is located a distance of 8 lightyears from the Earth. The astronaut measures the time of the one-way journey to be 6 yr. If the spaceship moves at a constant speed of 0.8c, how can the 8-ly distance be reconciled with the 6-yr trip time measured by the astronaut?

Solution: The 8 light-year distance is that measured according to the stationary observers, viz., the earthlings. According to the astronaut, who is in motion relative to Earth and Sirius, the distance is shortened by a factor γ:

$$L_{\text{astronaut}} = \frac{1}{\gamma} L_{\text{earthlings}} = (8\,\text{ly}) \left(\sqrt{1-(0.8c)^2/c^2}\right) = (8\,\text{ly})\,(0.6) = 4.8\,\text{ly} \tag{48}$$

The astronaut measures the trip to take 6 yr, which means the astronaut would report a velocity of

$$v = \frac{4.8\,\text{ly}}{6\,\text{yr}} = 0.8c \tag{49}$$

Thus, there is no paradox: the difference in measured times is due to time dilation/length contraction. More to the point: we can't divide one observers distance by another observer's time and expect to get sensible answers unless they are in the same reference frame!

9. A proton is accelerated to a velocity $v = 0.999c$ and sent down an evacuated metal tube 100 m long. Take the speed of light as $c = 3.0 \times 10^8$ m/s.

(a) In the protons reference frame, how long is the tube? **(b)** In the protons frame, how long does it take to traverse the length of the tube? **(c)** In the laboratory frame, how long does it take for the proton to traverse the length of the tube?

Solution: Since the proton is moving with respect to the tube, it will appear shortened from the proton's point of view by a factor γ from its proper rest length of $L = 100$ m:

$$\gamma = \frac{1}{\sqrt{1-0.999^2}} \approx 22.4 \tag{50}$$

$$L'_{\text{proton}} = \frac{L}{\gamma} = \frac{100\,\text{m}}{\gamma} \approx 4.46\,\text{m} \tag{51}$$

In the proton's rest frame, we would find the time taken by dividing the distance according to the proton by the velocity according to the proton. The proton sees the tube as having a distance L'_{proton}, and travel at velocity $v = 0.999c$ relative to the tube:

$$t'_{\text{proton}} = \frac{L'_{\text{proton}}}{v} \approx 1.48 \times 10^{-8}\,\text{s} = 14.8\,\text{ns} \tag{52}$$

In the laboratory frame, the proton travels at $0.999c$ and must traverse 100 m, so the time elapsed is

$$t_{\text{proton}} = \frac{L}{v} \approx 3.34 \times 10^{-7}\,\text{s} = 0.334\,\mu\text{s} \tag{53}$$

10. An atomic clock aboard a spaceship runs slow compared to an Earth-based atomic clock at a rate of 2.0 seconds per day. What is the speed of the spaceship?

Solution: The proper time t_p is that measured on earth, while a dilated time $t = \gamma t_p$ is measured on the ship. If the clock aboard the ship is 2 s per day slow, then using 1 day = 86400 s

$$t - t_p = \gamma t_p - t_p = (\gamma - 1)\,t_p = (\gamma - 1)\,(86400\,\text{s}) = 2\,\text{s} \tag{54}$$

This gives

$$\gamma = \frac{2}{86400} + 1 = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1.00002315 \tag{55}$$

Solving for v, we find $v \approx 0.0068c$.