Problem Set 3: Solutions

1. In an experiment to find the value of h, light at wavelengths 218 and 431 nm were shone on a clean sodium surface. The potentials that stopped the fastest photoelectrons were 5.69 and 0.59 V, respectively. What values of h and W, the sodium work function, are deduced?

Solution: The stopping potential can be related to the frequency f of incident light and the metal's work function W:

$$e\Delta V_{\text{stop}} = hf - W \tag{1}$$

In terms of wavelength, we know $\lambda f = c$, so

$$e\Delta V_{\text{stop}} = \frac{hc}{\lambda} - W \tag{2}$$

We are given two sets of data, pairs of stopping potentials and incident wavelengths: . That gives us two equations and two unknowns:

$$5.69\,\text{eV} = \frac{hc}{218\,\text{nm}} - W \tag{3}$$

$$0.59\,\text{eV} = \frac{hc}{431\,\text{nm}} - W \tag{4}$$

We wish to solve or h and W, with c being a known constant. The clever thing to do is subtract these two equations, that isolates h.

$$5.10\,\text{eV} = h\left(\frac{c}{218\,\text{nm}} - \frac{c}{431\,\text{nm}}\right) \tag{5}$$

$$\Longrightarrow \qquad h = 7.5 \times 10^{-15}\,\text{eV}\cdot\text{s} = 1.2 \times 10^{-38}\,\text{J}\cdot\text{s} \tag{6}$$

Plugging our value of h back into either of the two preceding equations gives

$$W = 6.98 \times 10^{-19}\,\text{J} = 4.36\,\text{eV} \tag{7}$$

This is not a particularly good experiment - the sodium work function is more like 2.28 eV[i], and the accepted value for Planck's constant is about half what this experiment finds.

2. A 0.3 MeV X-ray photon makes a "head on" collision with an electron initially at rest. Using conservation of energy and momentum, find the recoil velocity of the electron. Check your result with the Compton formula.

[i] http://hyperphysics.phy-astr.gsu.edu/hbase/tables/photoelec.html

Solution: Let E and E' be the initial and final energies of the photon, respectively. Conservation of energy then gives:

$$E = E' + (\gamma - 1)\, mc^2 \tag{8}$$

where the second term on the right is the electron's kinetic energy. For a head-on collision, the photon will recoil in the opposite direction, and the electron along the photon's original direction. Conservation of momentum then yields

$$\frac{E}{c} = -\frac{E'}{c} + \gamma m v \tag{9}$$

Writing the two equations together, after multiplying the second by c,

$$E - E' = (\gamma - 1)\, mc^2 \tag{10}$$

$$E + E' = \gamma m v c \tag{11}$$

Adding these two together allows us to eliminate E':

$$2E = (\gamma - 1)\, mc^2 + \gamma m v c = mc^2 \left(\frac{1}{\sqrt{1 - v^2/c^2}} + \frac{v/c}{\sqrt{1 - v^2/c^2}} - 1 \right) \tag{12}$$

Now we just solve for v

$$\frac{2E}{mc^2} = \frac{1 + v/c}{\sqrt{1 - v^2/c^2}} - 1 = \frac{1 + v/c}{\sqrt{(1 - v/c)\,(1 + v/c)}} - 1 = \sqrt{\frac{1 + v/c}{1 - v/c}} - 1 \tag{13}$$

$$\frac{1 + v/c}{1 - v/c} = \left(1 + \frac{2E}{mc^2}\right)^2 \tag{14}$$

$$\frac{v}{c} = \frac{\left(1 + 2E/mc^2\right)^2 - 1}{(1 + 2E/mc^2)^2 + 1} \approx 0.65 \tag{15}$$

How to check this with the Compton formula? You have the incident electron's energy, and thus its wavelength. You could use the Compton formula to get the exiting photon's wavelength, from which you could get its energy. The difference between incident and exiting photons' wavelengths is the electron's kinetic energy, which must be $(\gamma - 1)\, mc^2$. Given γ, you can find v for the electron.

3. A cavity is maintained at a temperature of 1650 K. At what rate does energy escape from the interior of the cavity through a hole in its wall of diameter 1.00 mm?

Solution: The rate of energy escape is just the emitted power. The total emitted power *per unit area* over all wavelengths is given by the StefanBoltzmann law: $I = \sigma T^4$. Given a temperature of

1650 K,

$$I = \sigma T^4 = \left(5.67 \times 10^8\,\mathrm{W/m^2 \cdot K^4}\right)(1650\,\mathrm{K})^4 \approx 4.2 \times 10^5\,\mathrm{W/m^2} \tag{16}$$

This power per unit area is emitted uniformly over the area of the hole, so the total power is:

$$P = IA = \left(4.2 \times 10^5\,\mathrm{W/m^2}\right)(\pi)(0.0005\,\mathrm{m})^2 \approx 0.33\,\mathrm{W} \tag{17}$$

4. Radio waves have a frequency of the order of 1 to 100 MHz. **(a)** What is the range of energies of these photons? **(b)** Our bodies are continuously bombarded by these photons. Why are they not dangerous to us?

Solution: **(a)** The energy per photon is given by $E = hf$. Thus

$$E_{\text{low}} = hf_{\text{low}} = \left(6.626 \times 10^{-34}\,\mathrm{J \cdot s}\right)\left(1 \times 10^6\,\mathrm{s^{-1}}\right) \approx 6.626 \times 10^{-28}\,\mathrm{J} = 4.141 \times 10^{-9}\,\mathrm{eV} \tag{18}$$

$$E_{\text{high}} = hf_{\text{high}} = \left(6.626 \times 10^{-34}\,\mathrm{J \cdot s}\right)\left(1 \times 10^8\,\mathrm{s^{-1}}\right) \approx 6.626 \times 10^{-26}\,\mathrm{J} = 4.141 \times 10^{-7}\,\mathrm{eV} \tag{19}$$

(b) These photons of negligible energy compared to the binding energies of molecules in our body, which are of the order of one to a few electron volts. Their energy is negligible compared to the thermal energy every atom in our bodies is already subject to, or the energy photons in ordinary sunlight.

5. An atom absorbs a photon of wavelength 375 nm and immediately emits another photon of wavelength 580 nm. What is the net energy absorbed by the atom in this process?

Solution: We just need to calculate the difference in energy between the incident and emitted photons, that must be the energy absorbed by the electron. It is useful to note that $hc \approx 1240\,\mathrm{eV \cdot nm}$ in finding the numerical result.

$$E_{\text{e-}} = hf_i - hf_a = \frac{hc}{\lambda_i} - \frac{hc}{\lambda_a} \approx 1240\left(\frac{1}{375} - \frac{1}{580}\right)\,\mathrm{nm} \approx 1.17\,\mathrm{eV} \tag{20}$$

The incident photon has an energy of about 3.3 eV, and the electron absorbed just over a third of it.

6. The Compton shift in wavelength $\Delta\lambda$ is independent of the incident photon energy $E_i = hf_i$. However, the Compton shift in *energy*, $\Delta E = E_f – E_i$ is strongly dependent on E_i. Find the expression for ΔE. Compute the fractional shift in energy for a 10 keV photon and a 10 MeV photon, assuming a scattering angle of 90°.

Solution: The energy shift is easily found from the Compton formula with the substitution $\lambda = hc/E$:

$$\lambda_f - \lambda_i = \frac{hc}{E_f} - \frac{hc}{E_i} = \frac{h}{mc}\left(1-\cos\theta\right) \tag{21}$$

$$\frac{cE_i - cE_f}{E_iE_f} = \frac{1-\cos\theta}{mc} \tag{22}$$

$$\Delta E = E_i - E_f = \left(\frac{E_iE_f}{mc^2}\right)\left(1-\cos\theta\right) \tag{23}$$

$$\frac{\Delta E}{E_i} = \left(\frac{E_f}{mc^2}\right)\left(1-\cos\theta\right) \tag{24}$$

Thus, the fractional energy shift is governed by the photon energy relative to the electron's rest mass, as we might expect. In principle, this is enough: one can plug in the numbers given for E_i and θ, solve for E_f, and then calculate $\Delta E/E_i$ as requested. This is, however, inelegant. One should really solve for the fractional energy change symbolically, being both more elegant and enlightening in the end. Start from Eq. 24 isolate E_f:

$$\frac{E_i - E_f}{E_i} = 1 - \frac{E_f}{E_i} = \frac{E_f}{mc^2}\left(1-\cos\theta\right) \tag{25}$$

$$1 = E_f\left[\frac{1}{E_i} + \frac{1}{mc^2}\left(1-\cos\theta\right)\right] \tag{26}$$

$$E_f = \frac{1}{1/E_i + \left(1-\cos\theta\right)/mc^2} = \frac{mc^2E_i}{mc^2 + E_i\left(1-\cos\theta\right)} \tag{27}$$

Now plug that back into the expression for ΔE we arrived at earlier, Eq. 24:

$$\frac{\Delta E}{E_i} = \left(\frac{1}{mc^2}\right)\left(\frac{mc^2E_i}{mc^2 + E_i\left(1-\cos\theta\right)}\right)\left(1-\cos\theta\right) \tag{28}$$

$$\frac{\Delta E}{E_i} = \frac{E_i\left(1-\cos\theta\right)}{mc^2 + E_i\left(1-\cos\theta\right)} = \frac{\left(\frac{E_i}{mc^2}\right)\left(1-\cos\theta\right)}{1 + \left(\frac{E_i}{mc^2}\right)\left(1-\cos\theta\right)} \tag{29}$$

This is even more clear (hopefully): Compton scattering is strongly energy-dependent, and the relevant energy scale is set by the ratio of the incident photon energy to the rest energy of the electron, E_i/mc^2. If this ratio is large, the fractional shift in energy is large, and if this ratio is small, the fractional shift in energy becomes negligible. Only when the incident photon energy is an appreciable fraction of the electron's rest energy is Compton scattering significant. The numerical values required can be found most easily by noting that the electron's rest energy is $mc^2 = 511\,\text{keV}$, which means we don't need to convert the photon energy to joules. One should find:

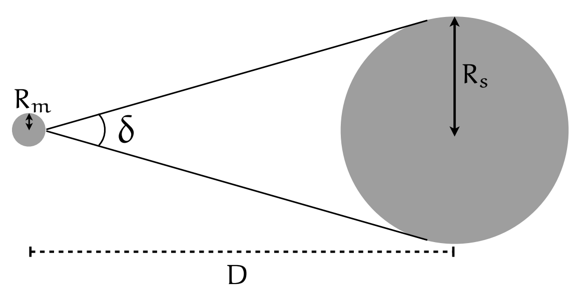
$$\frac{\Delta E}{E_i} \approx 0.02 \qquad 10\,\text{keV incident photon, } \theta = 90° \tag{30}$$

$$\frac{\Delta E}{E_i} \approx 0.95 \qquad 10\,\text{MeV incident photon, } \theta = 90° \tag{31}$$

Consistent with our symbolic solution, for the 10 keV photon the energy shift is negligible, while for the 10 MeV photon it is extremely large. Conversely, this means that the electron acquires a much more significant kinetic energy after scattering from a 10 MeV photon compared to a 10 keV photon.

7. Assume the sun radiates like a black body at 5500 K. Assume the moon absorbs all the radiation it receives from the sun and reradiates an equal amount of energy like a black body at temperature T. The angular diameter of the sun seen from the moon is about 0.01 rad. What is the equilibrium temperature T of the moon's surface? (Note: you do not need any other data than what is contained in the statement above.

Solution: The geometry of the problem is shown below, where δ is the angular diameter, R_m the moon's radius, R_s the sun's radius, and D the sun-moon distance.



The *definition* of angular diameter[ii], using the distances in the figure above, is

$$\tan\frac{\delta}{2} = \frac{R_s}{D} \tag{32}$$

With geometry in hand, we now need to balance the sun's power received by the moon with the power that the moon will re-radiate by virtue of its being at temperature T_m. Any body at temperature T emits a power $P = \sigma T^4 A$, where A is the area over which the radiation is emitted

[ii]See, e.g., `http://en.wikipedia.org/wiki/Angular_diameter`

and σ is a constant. Thus, since the sun emits radiation over its whole surface area $4\pi R_s^2$,

$$P_s = \sigma T_s^4 \left(4\pi R_s^2\right) \tag{33}$$

At a distance D corresponding to the moon's position, this power is spread over a sphere of radius D and surface area $4\pi D^2$. The amount of power the moon receives just depends on the ratio its absorbing area to the total area over which the power is spread out. The moon absorbs radiation over an area corresponding to its cross section, πR_m^2, so the fraction of the sun's total power that the moon receives is $\pi R_m^2/4\pi D^2$. Thus, the moon receives a power

$$P_{mr} = P_s \frac{\pi R_m^2}{4\pi D^2} = P_s \frac{R_m^2}{4D^2} = \sigma T_s^4 \left(4\pi R_s^2\right) \frac{R_m^2}{4D^2} \tag{34}$$

Absorbing this radiation from the sun will cause the moon to heat up to temperature T_m, and it will re-emit radiation as a black body at temperature T_m. Though the moon absorbs over its cross-sectional area, it emits over its whole surface area, so its emitted power is

$$P_{me} = \sigma T_m^4 \left(4\pi R_m^2\right) \tag{35}$$

Equilibrium requires that the power the moon receives equal the power the moon emits, so

$$P_{mr} = P_{me} \tag{36}$$

$$\sigma T_s^4 \left(4\pi R_s^2\right) \frac{R_m^2}{4D^2} = \sigma T_m^4 \left(4\pi R_m^2\right) \tag{37}$$

$$T_s^4 \frac{R_s^2}{4D^2} = T_m^4 \tag{38}$$

$$T_m = T_s \sqrt{\frac{R_s}{2D}} = T_s \sqrt{\frac{1}{2} \tan \frac{\delta}{2}} \approx 275\,\text{K} \tag{39}$$

Compare this with a mean lunar surface temperature at the equator of 220 K – not bad given the approximate geometry, and complete ignorance of reflection! It is interesting to see that the moon's radius does not factor in at all – it determines both the absorbed and emitted power in exactly the same way, and ends up canceling out.

8. Presume the surface temperature of the sun to be 5500 K, and that it radiates approximately as a blackbody. What fraction of the sun's energy is radiated in the visible range of $\lambda = 400 - 700\,\text{nm}$? One valid solution is to plot the energy density on graph paper and find the result numerically.

Solution: The emitted power per unit area per unit wavelength for a blackbody is given in a previous problem:

$$I(\lambda, T) = \frac{8\pi hc^2}{\lambda^5} \left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} \tag{40}$$

The power per unit area emitted over a range of wavelengths λ_1 to λ_2 is found by integrating $I(\lambda, T)$ over those limits, and the total power is integrating over all wavelengths from 0 to ∞. The fraction we desire is then the power over wavelengths λ_1 to λ_2 divided by the total power:

$$f = (\text{fraction}) = \frac{\int\limits_{\lambda_1}^{\lambda_2} I(\lambda, T)\, d\lambda}{\int\limits_0^{\infty} I(\lambda, T)\, d\lambda} \tag{41}$$

Let us first worry about the indefinite integral and put it in a bit simpler form.

$$\int I(\lambda, T)\, d\lambda = \int \frac{8\pi hc^2}{\lambda^5} \left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} d\lambda \tag{42}$$

It is convenient to make a change of variables to

$$u = \frac{hc}{\lambda k_b T} \qquad \text{or} \qquad \lambda = \frac{hc}{u k_b T} \tag{43}$$

This substitution implies

$$du = \frac{hc}{k_b T}\left(\frac{-d\lambda}{\lambda^2}\right) = -\frac{hc}{k_b T}\left(\frac{k_b T u}{hc}\right)^2 d\lambda = -\frac{u^2 k_b T}{hc} d\lambda \tag{44}$$

$$d\lambda = -\frac{hc}{u^2 k_b T} du \tag{45}$$

Performing the substitution,

$$\int I(\lambda, T)\, d\lambda = \int \frac{8\pi hc^2}{\lambda^5} \left[e^{\frac{hc}{\lambda k_b T}} - 1\right]^{-1} d\lambda = \int \frac{8\pi hc^2 u^5 k_b^5 T^5}{h^5 c^5} \frac{1}{e^u - 1} \frac{-hc}{u^2 k_b T} du \tag{46}$$

$$= -\frac{8\pi k_b^4 T^4}{h^3 c^2} \int \frac{u^3}{e^u - 1} du \tag{47}$$

The overall constants multiplying the integral will cancel in the fraction we wish to find:

$$f = \frac{\frac{8\pi k_b^4 T^4}{h^3 c^2} \int\limits_{u_1}^{u_2} \frac{u^3}{e^u - 1} du}{\frac{8\pi k_b^4 T^4}{h^3 c^2} \int\limits_0^{\infty} \frac{u^3}{e^u - 1} du} = \frac{\int\limits_{u_1}^{u_2} \frac{u^3}{e^u - 1} du}{\int\limits_{\infty}^{0} \frac{u^3}{e^u - 1} du} \tag{48}$$

Here the new limits of integration for the numerator are $u_1 = \frac{hc}{\lambda_1 k_b T} \approx 6.55\,\text{m}^{-1}$ and $u_2 = \frac{hc}{\lambda_1 k_b T} \approx$

$3.74\,\mathrm{m}^{-1}$, and the denominator has limits of ∞ and 0 after the substitution.

$$f = \frac{\int\limits_{6.55}^{3.74} \frac{u^3}{e^u-1}\,du}{\int\limits_{\infty}^{0} \frac{u^3}{e^u-1}\,du} \tag{49}$$

As it turns out, the integral in the denominator is known, and has a numerical value of $\pi^4/15$. The integral in the numerator has no closed-form solution, and must be found numerically. One thing we notice is that the denominator contains a factor $e^u - 1$, and at the limits of integration we have

$$e^{3.74} \approx 42 \tag{50}$$
$$e^{6.55} \approx 700 \tag{51}$$

In this case, since $e^u \gg 1$, to a good approximation we can write

$$\frac{1}{e^u - 1} \approx \frac{1}{e^u} = e^{-u} \tag{52}$$

The error we make in this approximation is in the worst case of order $1/43 \sim 2\%$ This makes the integral in the numerator of our fraction a known one, which can be integrated by parts:

$$\int\limits_{6.55}^{3.74} \frac{u^3}{e^u - 1}\,du \approx \int\limits_{6.55}^{3.74} u^3 e^{-u}\,du = e^{-u}\left(u^3 + 3u^2 + 6u + 6\right)\Big|_{6.55}^{3.74} \approx 2.29 \tag{53}$$

Thus,

$$f \approx \frac{2.29}{\pi^4/15} \approx 0.35 \tag{54}$$

About 35% of the sun's radiation should be in the visible range.[iii] A more exact numerical calculation gives closer to 36%, meaning our approximation above was indeed accurate to about 2%. One can do the "exact" numerical calculation from the start with Wolfram Alpha, for example, rather than resorting to approximations like we have here.

9. Show that it is impossible for a photon striking a free electron to be absorbed and not scattered.

Solution: All we really need to do is conserve energy and momentum for photon absorption by

[iii] This is what *leaves the sun*, to figure out what reaches the earth's surface we would have to account for reflection and absorption by the atmosphere. The fraction of visible light is closer to 42% at the earth's surface; see `uvb.nrel.colostate.edu/UVB/publications/uvb_primer.pdf` for example.

a stationary, free electron and show that something impossible is implied. Before the collision, we have a photon of energy hf and momentum h/λ and an electron with rest energy mc^2. Afterward, we have an electron of energy $(\gamma-1)+mc^2=\sqrt{p^2c^2+m^2c^4}$ (i.e., the afterward the electron has acquired kinetic energy, but retains its rest energy) and momentum $p_e=\gamma m v$. Momentum conservation dictates that the absorbed photon's entire momentum be transferred to the electron, which means it must continue along the same line that the incident photon traveled. This makes the problem one dimensional, which is nice.

Enforcing conservation of energy and momentum, we have:

$$\begin{aligned}
\text{(initial)} &= \text{(final)} && (55)\\
hf+mc^2 &= \sqrt{p^2c^2+m^2c^4} \qquad \text{energy conservation variant 1} && (56)\\
hf+mc^2 &= (\gamma-1)\,mc^2 \qquad \text{energy conservation variant 2} && (57)\\
\frac{h}{\lambda} &= p_e = \gamma m v \qquad \text{momentum conservation} && (58)
\end{aligned}$$

From this point on, we can approach the problem in two ways, using either expression for the electron's energy. We'll do both, just to give you the idea. First, we use conservation of momentum to put the electron momentum in terms of the photon frequency:

$$\frac{h}{\lambda}=p_e \qquad \Longrightarrow \qquad \frac{hc}{\lambda}=hf=p_e c \tag{59}$$

Now substitute that in the first energy conservation equation to eliminate p_e, square both sides, and collect terms:

$$\left(hf+mc^2\right)^2=\left(\sqrt{p^2c^2+m^2c^4}\right)^2=\left(\sqrt{h^2f^2+m^2c^4}\right)^2 \tag{60}$$

$$h^2f^2+2hfmc^2+m^2c^4=h^2f^2+m^2c^4 \tag{61}$$

$$2hfmc^2=0 \qquad \Longrightarrow \qquad f=0 \qquad \Longrightarrow \qquad p_e=v=0 \tag{62}$$

Thus, we conclude that the only way a photon can be absorbed by the stationary electron is if its frequency is zero, *i.e., if there is no photon to begin with!* Clearly, this is silly.

We can also use the second variant of the conservation of energy equation along with momentum conservation to come to an equally ridiculous conclusion:

$$hf = \frac{hc}{\lambda} = (\gamma - 1)\, mc^2 \qquad \text{energy conservation variant 2} \tag{63}$$

$$\frac{h}{\lambda} = \gamma m v \qquad \text{or} \qquad \frac{hc}{\lambda} = \gamma m v c \qquad \text{momentum conservation} \tag{64}$$

$$\Longrightarrow \qquad \gamma m v c = (\gamma - 1)\, mc^2 \tag{65}$$

$$(\gamma - 1)\, c = \gamma v \tag{66}$$

$$\frac{\gamma - 1}{\gamma} = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \text{(definition of } \gamma\text{)} \tag{67}$$

$$\left(\frac{\gamma - 1}{\gamma}\right)^2 = 1 - \frac{1}{\gamma^2} \tag{68}$$

$$\gamma^2 - 2\gamma + 1 = \gamma^2 - 1 \tag{69}$$

$$\gamma = 1 \qquad \Longrightarrow \qquad v = 0 \tag{70}$$

Again, we find an electron recoil velocity of zero, implying zero incident photon frequency, which means there is no photon in the first place! Conclusion: stationary electrons *cannot* absorb photons, but they can Compton scatter them.