Problem Set 4: "Whither, thou turbid wave" SOLUTIONS

Question zero is probably where the name of the problem set came from: "Whither, thou turbid wave?" It is from a Longfellow poem, *The Wave*, which you can find here: `http://www.litscape.com/author/Henry_Wadsworth_Longfellow/The_Wave.html`.

1. The wavefunction of a transverse wave on a string is

$$\psi(x,t) = (30.0\,\mathrm{cm})\cos\left[(6.28\,\mathrm{rad/m})\,x - (20.0\,\mathrm{rad/s})\,t\right] \tag{1}$$

Compute the frequency, wavelength, period, amplitude, phase velocity, and direction of motion.

Solution: The wavefunction has the form

$$\psi(x,t) = A\sin\left(kx - \omega t + \delta\right) \tag{2}$$

which lets us immediately identify

$$\omega = 2\pi f = 20\,\mathrm{rad/s} \qquad \Longrightarrow \qquad f = \frac{2\pi}{\omega} = \frac{10}{\pi}\,\mathrm{s}^{-1} \tag{3}$$

$$k = \frac{2\pi}{\lambda} = 6.28\,\mathrm{rad/m} \qquad \Longrightarrow \qquad \lambda = \frac{2\pi}{k} \approx 1.0\,\mathrm{m} \tag{4}$$

$$A = 30\,\mathrm{cm} \tag{5}$$

$$v = \frac{\omega}{k} \approx 3.18\,\mathrm{m/s} \tag{6}$$

Since the argument has the form $kx-\omega t$, the wave is traveling along the $+x$ direction.

2. What is the uncertainty in the location of a photon of wavelength 300 nm if this wavelength is known to an accuracy of one part in a million?

Solution: The momentum of the photon is

$$p = \frac{h}{\lambda} \tag{7}$$

The uncertainty in the photon momentum can be related to the uncertainty in wavelength (as we have done before with wavelength and energy uncertainty).

$$\Delta p = \left|\frac{\partial p}{\partial \lambda}\right|\Delta\lambda = \frac{h}{\lambda^2}\Delta\lambda = \frac{p\Delta\lambda}{\lambda} \tag{8}$$

Using the uncertainty principle,

$$\Delta x \geqslant \frac{\hbar}{2\Delta p} = \frac{\hbar\lambda}{2p\Delta\lambda} \approx 23.9\,\mathrm{mm} \tag{9}$$

This is a huge distance for a photon, but keep in mind that a photon takes only about 80 ps[i] to cover this distance!

3. Find the potential difference through which electrons must be accelerated (as in an electron microscope, for example) if we wish to resolve: (a) a virus of diameter 12 nm, (b) an atom of diameter 0.12 nm, (c) a proton of diameter 1.2 fm. Show your work, and do not forget about relativity.

Solution: The resolution of electron waves is roughly their wavelength. Wavelength we can find from momentum, from which we can get kinetic energy. Conservation of energy dictates that an electron accelerated from rest through a potential difference ΔV, thereby acquiring potential energy $e\Delta V$, will have a kinetic energy $K = e\Delta V$. Given that the smallest distance is well below the electron's Compton wavelength, we expect relativity may be important for at least that resolution, and we should perform a relativistic analysis.

An energy balance using the relativistic form for kinetic energy gives

$$K = e\Delta V = \sqrt{p^2c^2 + m^2c^4} - mc^2 \tag{10}$$

Solving for $p = h/\lambda$, we can relate wavelength and potential difference

$$\left(e\Delta V + mc^2\right)^2 = p^2c^2 + m^2c^4 \tag{11}$$

$$p^2 = \frac{1}{c^2}\left(e\Delta V + mc^2\right)^2 - m^2c^2 = \frac{h^2}{\lambda^2} \tag{12}$$

We can now find the potential difference in terms of the desired wavelength, which sets the microscope's resolution

$$\frac{h^2c^2}{\lambda^2} + m^2c^4 = \left(e\Delta V + mc^2\right)^2 \tag{13}$$

$$e\Delta V = \sqrt{\left(\frac{hc}{\lambda}\right)^2 + (mc^2)^2} - mc^2 \qquad \text{relativistic} \tag{14}$$

For simplicity in the resulting calculations, we can note $hc \approx 1240\,\mathrm{eV\cdot nm}$, $mc^2 \approx 511\,\mathrm{keV}$, and $1\,\mathrm{fm} = 10^{-6}\,\mathrm{nm}$. This way, putting the wavelength in nm we obtain $e\Delta V$ in electron volts, and the

[i] p = pico = 10^{-12}.

same numerical value is ΔV in volts. We can also note the classical result, which makes use of $K = p^2/2m$:

$$e\Delta V = \left(\frac{hc}{\lambda}\right)^2 \frac{1}{2mc^2} \qquad \text{classical} \tag{15}$$

wavelength (nm)	ΔV (relativistic)	ΔV (classical)
12	0.01 V	0.01 V
0.12	100 V	100 V
1.2×10^{-6}	10^9 V	10^{12} V

Even at 0.12 nm resolution the classical result is just fine, but for fm resolution, it breaks down badly.

4. In order to study the atomic nucleus, we would like to observe the diffraction of particles whose de Broglie wavelength is about the same size as the nuclear diameter, about 14 fm for a heavy nucleus such as lead. What kinetic energy should we use if the diffracted particles are (a) electrons? (b) Neutrons? (c) Alpha particles ($m = 4\,\text{u}$)?

Solution: We can make use of the result of the previous problem here – all we need to do is skip the step where we set $K = e\Delta V$ and solve for kinetic energy instead. That gives:

$$K = \sqrt{\left(\frac{hc}{\lambda}\right)^2 + (mc^2)^2} - mc^2 \tag{16}$$

particle	rest energy mc^2	K (relativistic)	K (classical)
electron	511 keV	88 MeV	7.7 GeV
neutron	940 MeV	4.2 MeV	4.2 MeV
α	3.73 GeV	1.1 MeV	1.1 MeV

The electron is clearly deeply in the relativistic limit, since the classical expression fails badly. One way to see this is that the kinetic energy implied far exceeds the rest mass of the electron, while for the other particles the kinetic energy is well below the rest mass. When the particle's energy is of order its rest mass, we expect relativistic effects to be important. Another rule of thumb is that the Compton wavelength $\lambda_c = h/mc$ for the particle in question is roughly the scale at which one has to worry about relativistic and quantum effects.

5. The speed of an electron is measured to within an uncertainty of 2.0×10^4 m/s. What is the size of the smallest region of space in which the electron can be confined?

Solution: Since $v/c \sim 10^{-4}$, we need not worry about relativity. Momentum is then just $p = mv$, and since m is a constant for an electron, we can say $\Delta p = m\Delta v$. We are given $\Delta v = 2.0 \times 10^4$ m/s, so we can just apply the uncertainty principle.

$$\Delta x \Delta p = \Delta x \left(m\Delta v\right) \geqslant \frac{h}{4\pi} \tag{17}$$

$$\implies \quad \Delta x = \frac{h}{4\pi m\Delta v} = \frac{\hbar}{2m\Delta v} \approx 2.8\,\text{nm} \tag{18}$$

6. Use the distribution of wave numbers

$$A(k) = A_o e^{-(k-k_o)^2/2(\Delta k)^2} \qquad k \in \mathbb{R} \tag{19}$$

and equation 4.23 from your text, viz.,

$$y(x) = \int A(k) \cos kx \, dk \tag{20}$$

to obtain equation 4.25 from your text, viz.,

$$y(x) = A_o \Delta k \sqrt{2\pi} e^{-(\Delta k\, x)^2/2} \cos k_o x \tag{21}$$

Solution: The mathematics will be far easier if we write the function $y(x)$ as a complex exponential and take the real part later:

$$y(x) = \int A(k) \cos kx \, dk = A(k) \Re\left\{e^{ikx}\right\}\, dk = A_o e^{-(k-k_o)^2/2(\Delta k)^2} \Re\left\{e^{ikx}\right\}\, dk \tag{22}$$

Remembering that we will take the real part in the end, integrating over all possible k gives

$$y(x) = \int\limits_{-\infty}^{\infty} A_o e^{-(k-k_o)^2/2(\Delta k)^2} e^{ikx} \, dk \tag{23}$$

Collect the exponential terms and expand, then factor out the parts that don't contain k (since they are not integrated)

$$y(x) = \int_{-\infty}^{\infty} A_o e^{-(k-k_o)^2/2(\Delta k)^2} e^{ikx}\,dk = A_o \int_{-\infty}^{\infty} e^{-\frac{1}{2(\Delta k)^2}\left[k^2-2kk_o+k_o^2-2ik(\Delta k)^2 x\right]}\,dk \tag{24}$$

$$= A_o \int_{-\infty}^{\infty} e^{-\frac{1}{2(\Delta k)^2}\left[k^2-2k\left(k_o+i(\Delta k)^2 x\right)\right]} e^{-\frac{k_o^2}{2(\Delta k)^2}}\,dk \tag{25}$$

Let $a = k_o + i\,(\Delta k)^2\,x$ for convenience, and note that the second exponential term can be pulled from the integral.

$$y(x) = A_o e^{-\frac{k_o^2}{2(\Delta k)^2}} \int_{-\infty}^{\infty} e^{-\frac{k^2-2ak}{2(\Delta k)^2}}\,dk \tag{26}$$

The part remaining to be integrated is *almost* a Gaussian, which we would know what to do with. We can make it a Gaussian by noting that $k^2 - 2ak = (k-a)^2 - a^2$.

$$y(x) = A_o e^{-\frac{k_o^2}{2(\Delta k)^2}} \int_{-\infty}^{\infty} e^{-\frac{(k-a)^2-a^2}{2(\Delta k])^2}}\,dk = A_o e^{-\frac{k_o^2}{2(\Delta k)^2}} \int_{-\infty}^{\infty} e^{\frac{2a^2}{2(\Delta k)^2}} e^{-\frac{(k-a)^2}{2(\Delta k])^2}}\,dk \tag{27}$$

Once again the second exponential term doesn't depend on k, so we may remove it from the integral. We will also substitute back in $a = k_o + i\,(\Delta k)^2\,x$:

$$y(x) = A_o e^{-\frac{k_o^2}{2(\Delta k)^2}} e^{\frac{\left(k_o+i(\Delta k)^2 x\right)^2}{2(\Delta k)^2}} \int_{-\infty}^{\infty} e^{-\frac{(k-a)^2}{2(\Delta k)^2}}\,dk \tag{28}$$

The integral is now just a Gaussian, which evaluates to $\Delta k\sqrt{2\pi}$. Expand what remains and simplify:

$$y(x) = A_o \Delta k\sqrt{2\pi}\, e^{\frac{1}{2(\Delta k)^2}\left[-k_o^2+k_o^2+2ik_o(\Delta k)^2 x-(\Delta k)^4 x^2\right]} = A_o \Delta k\sqrt{2\pi}\, e^{-x^2(\Delta k)^2/2} e^{ik_o x} \tag{29}$$

Now recall we only want the real part of the expression. The only complex quantity left is $e^{ik_o x}$, and $\Re\left\{e^{ik_o x}\right\} = \cos k_o x$. Thus we obtain the desired result:

$$y(x) = A_o \Delta k\sqrt{2\pi}\, e^{-x^2(\Delta k)^2/2} \cos k_o x \tag{30}$$

7. A particle of mass m is confined to a one-dimensional line of length L. From arguments based on the uncertainty principle, estimate the value of the smallest energy that the body can have.

Solution: The particle must be somewhere within the given segment, so the uncertainty in its position cannot be greater than L. If we say $\Delta x = L$, the uncertainty principle implies a momentum uncertainty of $\Delta p \geqslant h/4\pi L$. With maximum position uncertainty, we have a minimum momentum uncertainty. This in turn implies a minimum energy, since $K = p^2/2m$.

If we assume that the minimum momentum is just half of the uncertainty (i.e., the momentum may be zero plus or minus half the uncertainty), then $p_{min} = \frac{1}{2}\Delta p = h/8\pi L$. Thus,

$$K_{min} = \frac{p^2}{2m} = \frac{h^2}{128\pi^2 m L^2} \tag{31}$$

Given how crude our arguments are, the dependence on mass and length agrees reasonably well with the minimum energy for a particle in a one-dimensional potential well we derived in class,

$$E_1 = \frac{h^2}{8mL^2} \tag{32}$$

8. (a) Find the de Broglie wavelength of a nitrogen molecule in air at room temperature (293 K). (b) The density of air at room temperature and atmospheric pressure is 1.292 kg/m^3. Find the average distance between air molecules at this temperature and compare with the de Broglie wavelength. What do you conclude about the importance of quantum effects in air at room temperature? (c) Estimate the temperature at which quantum effects might become important.

Solution: **(a)** The de Broglie wavelength depends on the momentum, which depends on mass and velocity. The mass of a nitrogen molecule is 28 u or 4.65×10^{-26} kg. What its its average velocity? Neglecting vibrations of the molecule, each degree of freedom (axis of motion) gets on average $\frac{1}{2}k_B T$ worth of energy, for a total of $\frac{3}{2}k_B T$ average thermal energy. This shows up as kinetic energy of the molecule, so on average:

$$\frac{1}{2}m\langle v\rangle^2 = \frac{3}{2}k_B T \tag{33}$$

$$\implies \quad \langle v\rangle = \sqrt{\frac{3k_B T}{m}} \tag{34}$$

The de Broglie wavelength is then, on average,

$$\langle\lambda\rangle = \frac{h}{m\langle v\rangle} = \frac{h}{\sqrt{3k_B T m}} \tag{35}$$

At room temperature, this is about 0.03 nm.

(b) We don't even really need the density in fact. We know that at standard temperature and pressure, one mole of gas is about 6×10^{23} molecules and takes up 22.4 L (note $1\,\mathrm{L}=0.001\,\mathrm{m}^3$). This lets us figure out how many molecules there are per cubic meter:

$$\mathsf{N} = \text{molecules/m}^3 = \frac{6\times10^{23}\,\text{molecules}}{22.4\,\mathrm{L}}\,\frac{1\,\mathrm{L}}{0.001\,\mathrm{m}^3} \approx 2.7\times10^{25}\,\text{molecules/m}^3 \tag{36}$$

That means, on average, each molecule takes up

$$\mathsf{V} = \mathrm{m}^3/\text{molecule} \approx \frac{1\,\mathrm{m}^3}{2.7\times10^{25}\,\text{molecules}} \approx 3.7\times10^{-26}\,\mathrm{m}^3/\text{molecule} \tag{37}$$

If the molecules are distributed evenly, then there aught to be about $\sqrt[3]{\mathsf{N}}$ molecules along each side of a 1 m cube. Put another way, each one must on average occupy a little cube which is $\sqrt[3]{\mathsf{V}}$ on a side, giving the average spacing as

$$d \approx \sqrt[3]{\mathsf{V}} \approx 3\,\mathrm{nm} \tag{38}$$

One significant figure is plenty for an analysis of this nature. Using the given density, we could arrive at the average volume per molecule another way, using the molecular mass of nitrogen (28 g/mole):

$$\mathsf{V} = \left(\frac{1.292\,\mathrm{kg}}{\mathrm{m}^3}\right)\left(\frac{1\,\text{mole}}{28\,\mathrm{g}}\right)\left(\frac{1000\,\mathrm{g}}{1\,\mathrm{kg}}\right)\left(\frac{6\times10^{23}\,\text{molecules}}{1\,\text{mole}}\right) \approx 2.7\times10^{25}\,\text{molecules/m}^3 \tag{39}$$

(c) We would expect quantum effects to be important when the average de Broglie wavelength is about the same as the distance between molecules for a given density. Presuming we keep the density fixed, this means

$$\langle\lambda\rangle = \frac{h}{\sqrt{3k_BTm}} \sim d \tag{40}$$

$$\Longrightarrow \quad T \sim \frac{h^2}{3k_Bmd^2} \approx 0.02\,\mathrm{K} \tag{41}$$

9. Refer to the video lecture by Feynman you watched. (a) In the two slit experiment, is it possible to design an experiment which allows you to tell which slit an electron goes through without spoiling the interference pattern? (b) A particle can proceed from point A to point B along two indistinguishable paths. Path A has an amplitude of a, path B has an amplitude of b. What is the overall probability for the particle to go from point A to point B?

Solution: **(a)** No, it is not possible to design an experiment which allows one to tell which slit the electron went through without spoiling the interference. **(b)** When the paths are indistinguishable, we must add amplitudes and then square to get the probability, so $P=(a+b)^2$.

10. X-ray photons of wavelength 0.154 nm are produced by a copper source. Suppose that 1.00×10^{18} of these photons are absorbed by the target each second.

(a) What is the total momentum p transferred to the target each second? (b) What is the total energy E of the photons absorbed by the target each second? (c) If the beam shines perpendicularly onto a perfectly reflecting surface, what force does it exert on the surface? Recall $F = \Delta p/\Delta t$. (d) For these values, verify that the force on the target is related to the rate of energy transfer by

$$\frac{dp}{dt} = \frac{1}{c}\frac{dE}{dt}$$

Solution: **(a)** We know each photon has a momentum of h/λ, so the net momentum transfer in one second is the momentum per photon times the number of photons arriving in one second :

$$\frac{\Delta p_{\text{net}}}{\Delta t} = \left(1.00 \times 10^{18}\,\text{photons/s}\right)\left(\frac{h}{\lambda}\right) \approx 4.30 \times 10^{-6}\,\text{kg}\cdot\text{m/s} \tag{42}$$

(b) The energy of each photon is $E = hc/\lambda$, so the energy in one second is just the energy per photon times the number of photons arriving in one second:

$$\frac{\Delta E_{\text{tot}}}{\Delta t} = \left(1.00 \times 10^{18}\,\text{photons/s}\right)\left(\frac{hc}{\lambda}\right) = c\frac{\Delta p_{\text{net}}}{\Delta t} \approx 1.29 \times 10^{3}\,\text{J/s} = 1.29 \times 10^{3}\,\text{W} \tag{43}$$

(c) Since the photons are reflected, their momentum changes from p to $-p$, and the target must then acquire momentum $2p$ per photon that hits it to conserve momentum. That means the net force is just twice the net momentum transfer per second:

$$F_{\text{net}} = 2\frac{\Delta p_{\text{net}}}{\Delta t} = 8.60 \times 10^{-6}\,\text{N} \tag{44}$$

(d) Actually, this has been verified in part b already, but let's do it more formally. Let N be the number of photons striking the target, such that the rate of photons arriving is $dN/dt = 1.00 \times 10^{18}\,\text{s}^{-1}$. This gives 1.00×10^{18} photons arriving in one second as the problem specifies, we have only assumed that they arrive at a uniform rate.

$$\frac{dp_{\text{net}}}{dt} = \frac{d}{dt}\left(N\frac{h}{\lambda}\right) = \frac{h}{\lambda}\frac{dN}{dt} \tag{45}$$

$$\frac{dE_{\text{net}}}{dt} = \frac{d}{dt}\left(N\frac{hc}{\lambda}\right) = \frac{hc}{\lambda}\frac{dN}{dt} \tag{46}$$

$$\Longrightarrow \quad \frac{dp}{dt} = \frac{1}{c}\frac{dE}{dt} \tag{47}$$

We can also verify the relationship numerically,

$$\frac{\mathsf{dE}/\mathsf{dt}}{\mathsf{dp}/\mathsf{dt}} = \frac{1.29 \times 10^3 \text{ J/s}}{4.3 \times 10^{-6} \text{ kg} \cdot \text{m/s}} = 3.00 \times 10^8 \text{ m/s} = \mathsf{c} \tag{48}$$