University of Alabama
Department of Physics and Astronomy
PH 253-002 Spring 2019

Homework 3

Instructions:

1. Answer all questions below. Show your work for full credit.
2. All problems are due by 4:45pm on Fri 22 Feb as a hard copy, or by 11:59pm on Fri 22 Feb via Blackboard
3. You may collaborate, but everyone must turn in their own work.

1. **(a)** Consider a thermal neutron, that is, a neutron with speed v corresponding to the average thermal energy at the temperature $T = 300\,\mathrm{K}$. Is it possible to observe a diffraction pattern when a beam of such neutrons falls on a crystal? Justify your answer with a calculation. **(b)** In a large accelerator, an electron can be provided with an energy over $1\,\mathrm{GeV} = 10^9\,\mathrm{eV}$. What it is the de Broglie wavelength corresponding to such electrons?

2. Consider the wave function

$$\psi(x,t) = \left[Ae^{ipx/\hbar} + Be^{-ipx/\hbar}\right]e^{-ip^2t/2m\hbar} \tag{1}$$

Find the *probability current* j for this wave function. Probability current is defined by

$$j(x,t) = \frac{\hbar}{2mi}\left(\psi^*\frac{\partial\psi}{\partial x} - \frac{\partial\psi^*}{\partial x}\psi\right) \tag{2}$$

(Recall ψ^* is the complex conjugate of ψ.) The probability current describes the flow of probability (probability per unit area per unit time), and is analogous to mass or charge flow.

3. Consider a particle described by a wave function $\psi(x,t)$. Calculate the time derivative of the probability density, $\rho \equiv |\psi|^2$, and show that the continuity equation

$$\frac{\partial\rho(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0 \tag{3}$$

where j is as defined in the previous problem.

4. Use the ground-state wave function of the simple harmonic oscillator (below) to find $x_{\mathrm{av}} = \langle x\rangle$, $(x^2)_{\mathrm{av}} = \langle x^2\rangle$, and Δx. Use the normalization constant $a = (m\omega_o/\pi\hbar)^{1/4}$. *Hint: Recall what happens when you integrate odd functions over all x, and check integral tables for the even functions.*

5. *Zero point energy of a harmonic oscillator.* The frequency f of a harmonic oscillator of mass m and elasticity constant k is given by the equation

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \tag{4}$$

The energy of the oscillator is given by

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2 \tag{5}$$

where p is the system's linear momentum and x is the displacement from its equilibrium position. Use the uncertainty principle, $\Delta x \Delta p \approx \hbar/2$, to express the oscillator's energy E interms of x and show, by taking the derivative of this function and setting $dE/dx = 0$, that the minimum energy of the oscillator (its ground state energy) is $E_{\min} = hf/2$.

6. What is the minimum energy of a neutron confined to a region of space of nuclear dimensions $(1 \times 10^{-14}\,\mathrm{m})$?

7. The state of a free particle is described by the following wave function

$$\psi(x) = \begin{cases} 0 & x < -b \\ A & -b \leqslant x \leqslant 5b \\ 0 & x > 5b \end{cases} \tag{6}$$

(a) Determine the normalization constant A.
(b) What is the probability of finding the particle in the interval $[0, b]$?
(c) Determine $\langle x \rangle$ and $\langle x^2 \rangle$ for this state.
(d) Find the uncertainty in position $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$.

8. The time-dependent Schrödinger equation for a simple harmonic oscillator reads

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 \psi = E\psi \tag{7}$$

The ground state wave function has the form $\psi_o = a e^{-\alpha^2 x^2}$. Determine the value of the constant α and the energy of the state. (Your answer should match the result given in the textbook, the point is to prove it is correct.)

9. An electron is trapped in an infinitely deep one-dimensional well of width 0.17 nm. Initially the electron occupies the $n = 4$ state. **(a)** Suppose the electron jumps to the ground state with the accompanying emission of a photon. What is the energy of the photon? **(b)** Find the energies of other photons that could be emitted if the electron takes different paths to the ground state.

10. A quantum-mechanical duck lives in a world in which $h = 2\pi\,\mathrm{J{\cdot}s}$. The duck has a mass of 1.75 kg and is initially known to be within a pond 1.00 m wide. What is the minimum uncertainty in her speed?