1. a) does wavelength compare favorably with atomic spacing in crystals ($\sim 10^{-10}$ m)?

$\lambda = \frac{h}{p} = \frac{h}{mv}$ thermal speed $v \ll c$, relativity not needed

thermal speed: consv. E $K = \frac{1}{2}mv^2 = E_{th} = \frac{3}{2}k_BT$

$\Rightarrow v = \sqrt{\frac{3k_BT}{m}} \simeq 2.7\times10^3$ m/s classical expression justified

$v \ll c$

$\Rightarrow \lambda = \frac{h}{mv} = \frac{h}{m}\sqrt{\frac{m}{3k_BT}} = \frac{h}{\sqrt{3k_BTm}} \simeq 1.5\times10^{-10}$ m

$\lambda \sim$ same order as crystal spacing, it will work

b) $E = 1$ GeV $\gg mc^2$ for electron, will need relativity

$K = \sqrt{p^2c^2+m^2c^4} - mc^2$

$\Rightarrow p^2 = \frac{(K+mc^2)^2 - m^2c^4}{c^2}$

$\lambda = \frac{h}{p} \simeq 1.2\times10^{-15}$ m

of order H atom nucleus diameter

2. need $\frac{\partial \psi}{\partial x}$, ψ^*, $\frac{\partial \psi^*}{\partial x}$. presume A, B could be complex ...

$$\frac{\partial \psi}{\partial x} = \frac{ip}{\hbar}\left[A e^{ipx/\hbar} - B e^{-ipx/\hbar}\right] e^{-ip^2 t/2m\hbar}$$

$$\psi^* = \left[A^* e^{-ipx/\hbar} + B^* e^{ipx/\hbar}\right] e^{ip^2 t/2m\hbar}$$

$$\frac{\partial \psi^*}{\partial x} = \frac{ip}{\hbar}\left[-A^* e^{-ipx/\hbar} + B^* e^{ipx/\hbar}\right] e^{ip^2 t/2m\hbar}$$

$$\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} = \frac{ip}{\hbar}\left[AA^* - A^* B e^{-2ipx/\hbar} + AB^* e^{2ipx/\hbar} - BB^*\right]$$
$$- \frac{ip}{\hbar}\left[-AA^* - A^* B e^{-2ipx/\hbar} + AB^* e^{2ipx/\hbar} + BB^*\right]$$

$$\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} = \frac{ip}{\hbar}\left[2A^*A - 2B^*B\right] = \frac{2ip}{\hbar}\left[|A|^2 - |B|^2\right]$$

$$j = \frac{\hbar}{2mi} \cdot \frac{2ip}{\hbar}\left[|A|^2 - |B|^2\right] = \frac{p}{m}\left[|A|^2 - |B|^2\right] = v\left(|A|^2 - |B|^2\right)$$

with $p = mv$

[note $\rho = |\psi|^2 = |A|^2 - |B|^2$, and we have $j = \rho v$
like an ideal incompressible fluid]

3. $$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\psi^*\psi) = \frac{\partial \psi^*}{\partial t}\psi + \psi^*\frac{\partial \psi}{\partial t}$$

$$\frac{\partial \psi}{\partial t} = -\frac{\hbar}{2mi}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{i\hbar}V\psi$$ time-dep Schrödinger

$$\frac{\partial \psi^*}{\partial t} = \frac{\hbar}{2mi}\frac{\partial^2 \psi^*}{\partial x^2} - \frac{1}{i\hbar}V\psi^*$$ + complex conjugate

$$\Rightarrow \frac{\partial \rho}{\partial t} = -\frac{\hbar}{2mi}\left(\psi^*\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi^*}{\partial x^2}\right) + \frac{1}{i\hbar}\big(\underbrace{\psi^* V\psi - \psi V\psi^*}_{\emptyset}\big)$$

$$\frac{\partial \rho}{\partial t} = -\frac{\hbar}{2mi}\left(\psi^*\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi^*}{\partial x^2}\right)$$

$$\frac{\partial j}{\partial x} = \frac{\hbar}{2mi}\left[\frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x} + \psi^*\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2}\psi - \frac{\partial \psi^*}{\partial x}\frac{\partial \psi}{\partial x}\right]$$

$$= \frac{\hbar}{2mi}\left(\psi^*\frac{\partial^2 \psi}{\partial x^2} - \psi\frac{\partial^2 \psi}{\partial x^2}\right) = -\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial j}{\partial x} = 0$$ same as charge conservation or mass continuity for an ideal incompressible fluid!

4. $\psi = \sqrt[4]{\frac{m\omega_0}{\pi\hbar}}\, e^{-\alpha^2 x^2}$ $\qquad |\psi|^2 = \sqrt{\frac{m\omega_0}{\pi\hbar}}\, e^{-2\alpha^2 x^2}$

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi|^2\,dx = \int_{-\infty}^{\infty} \sqrt{\frac{m\omega_0}{\pi\hbar}}\, x e^{-2\alpha^2 x^2}\,dx = 0 \quad \text{odd function}$$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2|\psi|^2\,dx = \int_{-\infty}^{\infty} \sqrt{\frac{m\omega_0}{\pi\hbar}}\, \underbrace{x^2 e^{-2\alpha^2 x^2}}_{\text{lookup}}\,dx = \frac{1}{2}\sqrt{\frac{m\omega_0}{\pi\hbar}}\sqrt{\frac{\pi}{8\alpha^6}} = \frac{1}{4}\sqrt{\frac{m\omega_0}{2\hbar\alpha^6}}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{1}{2}\sqrt[4]{\frac{m\omega_0}{2\hbar\alpha^6}}$$

using $\alpha^2 = \frac{m\omega_0}{2\hbar}$ (problem 8)

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega_0}, \quad \Delta x = \sqrt{\frac{\hbar}{2m\omega_0}}$$

5. $$P_{min} = \Delta P_{min} = \frac{\hbar}{2\Delta x} = \frac{\hbar}{2X_{min}}$$

$$E = \frac{\hbar^2}{4X_{min}^2}\cdot\frac{1}{2m} + \frac{1}{2}kX_{min}^2 = \frac{\hbar^2}{8mX_{min}^2} + \frac{1}{2}kX_{min}^2 = \frac{\hbar^2}{8mX_{min}^2} + \frac{1}{2}m\omega_0^2 X_{min}^2$$

$$k = \omega_0^2 m$$

$$\frac{\partial E}{\partial X_{min}} = \frac{-\hbar^2}{4mX_{min}^3} + m\omega_0^2 X_{min} = 0 \Rightarrow 4m^2\omega_0^2 X_{min}^4 = \hbar^2$$

$$\Rightarrow X_{min}^2 = \frac{\hbar}{2m\omega_0}$$

$$\Rightarrow E_{min} = \frac{\hbar^2}{8m}\,\frac{2m\omega_0}{\hbar} + \frac{1}{2}m\omega_0^2\,\frac{\hbar}{2m\omega_0} = \frac{\hbar\omega_0}{4} + \frac{\hbar\omega_0}{4} = \frac{1}{2}\hbar\omega = \frac{1}{2}hf \checkmark$$

6. particle confined to a region with $L = 10^{-14}$ m
in 1D, particle in a box $E = \frac{\hbar^2}{8mL^2} \simeq 2$ MeV

7. $|\Psi|^2 = A^2$

a) $1 = \int_{-\infty}^{\infty} P(x)\,dx = \int_{-\infty}^{\infty} |\Psi|^2 dx = \int_{-b}^{5b} A^2 dx = 6bA^2 \Rightarrow A^2 = \frac{1}{6b}$

$|\Psi|^2 = 0$ outside $[-b, 5b]$

b) $P(\text{in } [0,b]) = \int_0^b |\Psi|^2 dx = \int_0^b A^2 dx = A^2 b = \frac{1}{6}$

Ψ is constant over interval of width $6b$, so probability of finding it in an interval of width b should be $\frac{1}{6}$

c) $\langle x \rangle = \int_{-b}^{5b} A^2 x\,dx = \frac{x^2}{2}\Big|_{-b}^{5b} A^2 = \left(\frac{25b^2 - b^2}{2}\right) A^2 = 12b^2 \cdot \frac{1}{6b} = 2b$

$\langle x^2 \rangle = \int_{-b}^{5b} A^2 x^2 dx = \frac{x^3}{3}\Big|_{-b}^{5b} A^2 = \frac{125b^3 - -b^3}{3} A^2 = \frac{126}{3} b^3 A^2$

$\langle x^2 \rangle = 42 b^3 A^2 = 7b^2$

d) $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{7b^2 - 4b^2} = b\sqrt{3}$

8. $\psi = a e^{-\alpha^2 x^2}$ normalize to find a

$$1 = \int_{-\infty}^{\infty} |\psi|^2 dx = a^2 \int_{-\infty}^{\infty} e^{-2\alpha^2 x^2} dx = \frac{\sqrt{\pi}}{\alpha} a^2 \Rightarrow \alpha^2 = \pi a^4$$ need more info...

use Schrödinger eqn.

$$\frac{\partial \psi}{\partial x} = -2\alpha^2 x a e^{-\alpha^2 x^2} \qquad \frac{\partial^2 \psi}{\partial x^2} = -2\alpha^2 a e^{-\alpha^2 x^2} + 4\alpha^4 x^2 a e^{-\alpha^2 x^2}$$

$$= (4\alpha^4 x^2 - 2\alpha^2)\psi$$

$$\Rightarrow E\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2} m\omega_0^2 x^2 \psi = -\frac{\hbar^2}{2m}(4\alpha^4 x^2 - 2\alpha^2)\psi + \frac{1}{2} m\omega_0^2 x^2 \psi$$

can only work for all x if constant terms and x^2 terms multiplying ψ are separately equal

$$\Rightarrow E = -\frac{\hbar^2}{2m}(-2\alpha^2) = \frac{\hbar^2 \alpha^2}{m}$$

$$\Rightarrow -\frac{\hbar^2}{2m}(4\alpha^4)x^2 + \frac{1}{2} m\omega_0^2 x^2 = 0$$

$$\frac{2\hbar^2\alpha^4}{m} = \frac{1}{2} m\omega_0^2 , \quad \alpha^4 = \frac{m^2\omega_0^2}{4\hbar^2}$$ (for all $x \neq 0$)

Combine: $E = \frac{\hbar^2\alpha^2}{m} = \frac{\hbar^2}{m}\frac{m\omega_0}{2\hbar} = \frac{1}{2}\hbar\omega_0$ (same as problem 5)

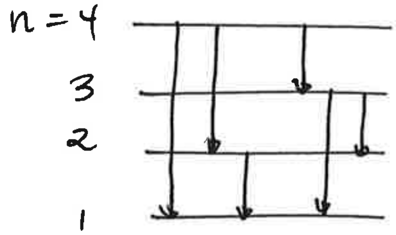
$$\alpha = \sqrt[4]{\frac{m^2\omega_0^2}{4\hbar^2}} = \sqrt{\frac{m\omega_0}{2\hbar}}$$

$$a^2 = \alpha\sqrt{\pi} , \quad a = \sqrt[4]{\pi}\sqrt{\alpha} = \sqrt[4]{\frac{\pi m\omega_0}{2\hbar}}$$ ✓

9. $L = 0.17\,nm = 1.7\times10^{-10}\,m$, $n = 4$

$$E_1 = \frac{h^2}{8mL^2} = 13.03\,eV \qquad E_n = n^2 E_1$$

starting from $n=4$, there are $\binom{4}{2} = 6$ possible transitions



transition	energy
$4 \to 1$	$E_4 - E_1 = (4^2-1^2)E_1 = 195.4\,eV$
$4 \to 2$	$E_4 - E_2 = (4^2-2^2)E_1 = 156.3\,eV$
$4 \to 3$	$E_4 - E_3 = (4^2-3^2)E_1 = 91.2\,eV$
$3 \to 2$	$E_3 - E_2 = (3^2-2^2)E_1 = 65.1\,eV$
$3 \to 1$	$E_3 - E_1 = (3^2-1^2)E_1 = 104.2\,eV$
$2 \to 1$	$E_2 - E_1 = (2^2-1^2)E_1 = 39.1\,eV$

photon emitted has energy equal to the difference in energy between initial and final levels. going from level n to m,

$$\Delta E = hf = E_n - E_m = \frac{n^2h^2}{8mL^2} - \frac{m^2h^2}{8mL^2} = (n^2-m^2)\frac{h^2}{8mL^2} = (n^2-m^2)E_1$$

10. $\Delta x \Delta p \geq \hbar/2$ $\quad p = mv$ so $\Delta p = m\Delta v$ if mass is known

$\hbar = h/2\pi = 1\,J\cdot s$ $\quad \Delta x = 1\,m$

$\Rightarrow m\Delta x\Delta v \geq \hbar/2$ or $\Delta v \geq \dfrac{\hbar}{2m\Delta x} \simeq 0.29\,m/s$