Radiation, Energy Quanta

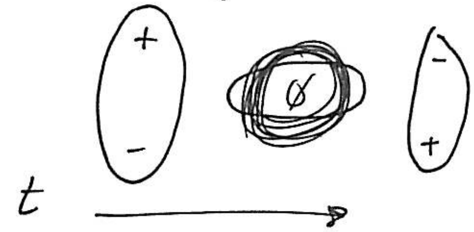
- Recall kinetic theory from mechanics or thermo
- system of large numbers of intercolliding atoms (e.g. gas)
  - identical, indistinguishable atoms
  - at temp T, mean KE is $\frac{1}{2}k_B T$ per degree freedom
    e.g. $\frac{3}{2}k_B T$ for single free atom / (x, y, z)
  - could derive equation of state - $PV = NkT$
  - could derive mean speed & distribution
- now we know light = particles = photons; oscill chgs radiate
  - can we figure out radiation similarly?
  - e.g., glow of hot objects?
  - answer is <u>NO</u>, but *why* reveals a lot!

- to start: try to figure out radiation classically
  see what's wrong.

$\boxed{\text{Model for thermal radiation}}$

- atoms ~ oscillators (OK... bonds... come back to this)
- oscillation of atoms due to thermal energy
  - $k_B T = \langle \frac{1}{2}mv^2 \rangle$ random direction
- so far, like ideal gas... larger T, more osc.

- what happens when an atom oscillate?
  - charge distribution accellerates / oscillates!



- an oscillating dipole
- accellerating charges RADIATE

- oscillating electron radiates light!

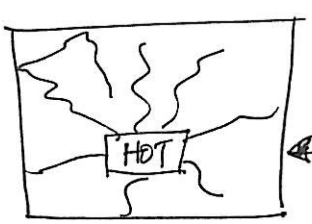
(1) electrons acquire thermal energy → kinetic energy / osc

(2) collisions → energy transfer → equilibrium

(3) oscillations → radiation loss → cooling

$\boxed{\text{Basically: hot objects radiate, and thereby cool}}$

- We *can* reach an equilibrium by putting the hot object in an insulated box, mirrored
- no light escapes box, energy contained ⇒ equil
  - all thermal & radiant energy stays in box



insulated mirrored box

- radiated light reabsorbed eventually

- if we can figure out radiation of an accelerated charge, we are GOOD

$\boxed{\text{Toy atom}}$: mass-spring (bond)

$m \ll M$

$\omega_0 \cong \sqrt{\frac{k}{m}}$

- presume electric forces dominate
- just a harmonic oscillator!
- presume proton very heavy, $\approx$ stationary

$$m\frac{d^2x}{dt^2} + m\omega_0^2 x = -eE(x,t) \quad \text{forces}$$

– if x oscillates, so does $\vec{E}$! Let $\underline{E = E_0 e^{i\omega t}}$

– then solution is known!

$$\vec{x} = \left(\frac{e}{m}\right)\frac{\vec{E}}{\omega_0^2-\omega^2}$$

(can add collision damping easily)

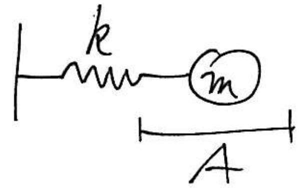
$$\vec{a} = -\omega^2\vec{x} = \left(\frac{e}{m}\right)\frac{\omega^2\vec{E}}{\omega_0^2-\omega^2}$$

So what? from accel. we get radiated power!

(Larmor formula)
$$P = \frac{e^2a^2}{6\pi\epsilon_0 c^3} = \frac{e^2x^2\omega^4}{6\pi\epsilon_0 c^3} = \frac{e^3E_0^2\,\omega^4}{6\pi\epsilon_0 m c^3(\omega_0^2-\omega^2)^2}$$

- most of rad at resonant freq
- ! $P \propto \omega^4$ ... DIVERGES!

Worse: energy of an oscillator?



$E = \frac{1}{2} k A^2$ $\quad$ A = max. ampl

$\omega_0 = \sqrt{\frac{k}{m}}$ or $k = \omega_0^2 m$

$\Rightarrow E = \frac{1}{2} m \omega^2 A^2$

$\boxed{\text{if each oscillator gets } \frac{3}{2} k_B T \text{ of energy,}}$

$\frac{3}{2} k_B T = \frac{1}{2} m \omega^2 A^2 \Rightarrow \sqrt{\frac{3 k_B T}{m A^2}} = \omega$

-or- as $T \uparrow$ ω diverges!

$\boxed{\text{SAME problem w/ ideal gas!}}$

Way out? "equipartition theorem"

i.e. equally-spaced levels

HACK ... but quantum mech. justifies

in the end, classically

$$I(\omega) = \text{rad emitted @ } \omega = \frac{\omega^2 k_B T}{\pi^2 c^2} = \frac{2\pi c k_B T}{\lambda^4}$$

Something is deeply wrong!

$\boxed{\text{Worse: atoms are not stable in this model}}$

NO bound ⊕ ⊖ system is! Always lose E by RAD

So the problem gets <u>worse</u>. From thermo we expect

$P \propto kT$ more thermal energy, more power

$P \propto \omega^4$ electrodynamics $\Bigg\} \Rightarrow I(\omega) \sim \dfrac{\omega^2 kT}{\pi^2 c^2}$

<u>Conclusions</u>: hot objects emit x-rays! <u>no "green hot"</u>

power diverges!

$\boxed{\text{fail}}$

$\boxed{\text{What is the problem?}}$

(*) we implicitly assumed all oscillating atoms get kT worth of energy each!

- can <u>only</u> diverge! Further... <u>atoms not stable</u>

$\boxed{\text{Resolution}}$ perhaps atoms cannot emit arbitrary amounts of energy? (Planck)

(*) Atoms "ONLY"* <u>radiate when they lose a multiple of fund. E</u>

that is: <u>what if energy only comes in discrete bundles?</u>

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e.g. vibrating string - resonant modes due to structure

perhaps it is not just matter that is discrete?

Specifically: what if oscillators only emit energy in bundles proportional to frequency? $\boxed{\text{Harmonics?}}$

e.g. oscillator: $E \propto \omega^2$ what if E has fixed values?

$\Delta E \propto f$ or $\Delta E = hf = \frac{h\omega}{2\pi} = \hbar\omega$

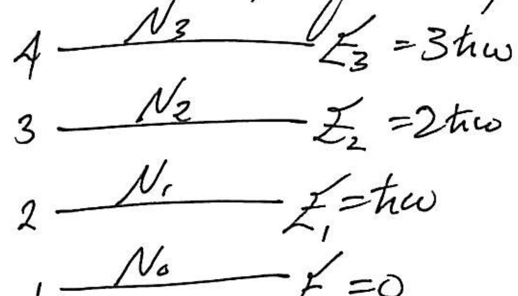
$\boxed{\text{TINY!}}$ (why we don't notice)

h = Planck's const $= 6.626 \times 10^{-34}$ J·s

$\hbar = h/2\pi$ $\boxed{\text{Low E to notice! prob? k small!}}$

- Start of quantum mechanics: quantized observables
- familiar: used to derive specific heat of gasses "equipartition theorem"
- $\boxed{\text{atoms} \Rightarrow \text{discrete matter; why not their energies?}}$
- justified by quantum physics! Implications?

- Suppose we have an oscillator (atom), resonant ω_0 (e.g. diatomic molecule, simple xtal)
- $\boxed{\text{Collection of many of these}}$
- let allowed energies be DISCRETE, equally spaced
  - spacing: $\hbar\omega$
  - label by n
  - let lowest be $\emptyset$
    fully occupied at $T=0$; fund mode
  - otherwise, N_n atoms in total have each energy E_n



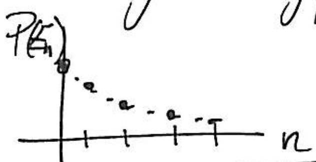
- if we have many atoms, all with vibrational freq ω_0
  - some will be in the lowest state
  - some in higher states
- atoms near $T=0$ will be unlikely to have enough thermal E to reach higher states
- as $T \uparrow$, more & more atoms on average can make it to higher states

? What is the energy, on average, of the oscillators

- We can use the Boltzmann factor to find the probability of finding an atom in a given state!

$P(E)$ = prob. of an atom having energy E

$$P(E) = (\text{const})\, e^{-E/k_B T}$$



(const): determined by normalization, i.e. sum of all probabilities $= 1$

: set by lowest energy

[we'll prove this later, used in thermo ...]

Since $E_0 = 0$,
in this case, (const $= 1$) $\quad P(E) = e^{-E/k_B T}$

- many oscillators, let's say N_{tot}
- infinity of levels $n = 0, 1, \cdots \infty$
- let the number in each level be $N_n = N_0, N_1, \cdots N_n$

energy is $E_n = E_0, E_1, E_2 \cdots E_n$

or $\boxed{E_n = 0, \hbar\omega, 2\hbar\omega, \ldots n\hbar\omega}$

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average energy? $\dfrac{(\text{total } E)}{(\#\text{atoms})} = \dfrac{\sum(\#\text{ per level})(\text{energy of level})}{(\#\text{ atoms})}$

for any level, occupation is $(\#\text{atoms})(\text{prob. occup.})$

or $N_n = N_0 e^{-E_n/k_BT}$ N_0 = number in lowest level

so $\left(N_1 = N_0 e^{-\hbar\omega/kT}, \; N_2 = N_0 e^{-2\hbar\omega/kT}, \; \ldots\right)$

$(T = 0, \text{ all in } N_0)$

$\boxed{\text{total energy is summing } \sum N_n E_n}$

$$E_{tot} = \sum_{n=0}^{\infty} N_n E_n = \sum_{n=0}^{\infty} \left(N_0 e^{-n\hbar\omega/k_BT}\right)(n\hbar\omega)$$

let $\boxed{x = e^{-\hbar\omega/k_BT}}$, then $N_n = N_0 x^n$

$$E_{tot} = N_0 \hbar\omega \sum_{n=0}^{\infty} n x^n = N_0 \hbar\omega \left(0 + x + 2x^2 + \ldots\right)$$

how many oscillators in total? add number in each level!

$$N_{tot} = \sum_{n=0}^{\infty}{}' N_n = N_0 \sum{}' x^n = N_0 (1 + x + x^2 + \cdots)$$

average energy is then

$$\langle E \rangle = \frac{E_{tot}}{N_{tot}} = \frac{N_0 \hbar\omega \sum_{n=0}^{\infty}{}' n x^n}{N_0 \sum_{n=0}^{\infty}{}' x^n} = \hbar\omega \left(\frac{\sum{}' n x^n}{\sum{}' x^n} \right)$$

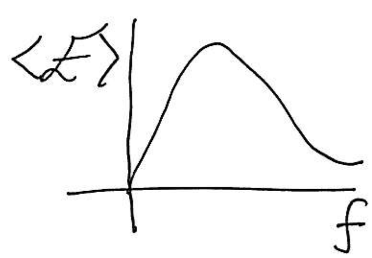
Both sums are known!

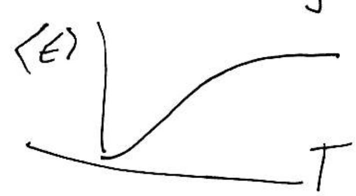
$$\sum_{i=0}^{\infty}{}' i x^i = \frac{x}{(1-x)^2} \qquad \sum_{i=0}^{\infty}{}' x^n = \frac{1}{1-x} \text{ (geometric)}$$

$$\Rightarrow \langle E \rangle = \hbar\omega \left(\frac{x}{(1-x)^2} \cdot \frac{1}{1/(1-x)} \right) = \hbar\omega \left(\frac{x}{1-x} \right)$$

recall $x = e^{-\hbar\omega/k_B T}$

$$\langle E \rangle = \hbar\omega \frac{e^{-\hbar\omega/k_B T}}{1 - e^{-\hbar\omega/k_B T}} = \underline{\underline{\frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1}}}$$





- first true quantum formula!
- <u>very</u> different from classical result, viz. $\langle E \rangle = k_B T$
- Does NOT suffer from divergences, etc.
  SATURATES @ high T

we can check the limits: need $\langle E \rangle \to kT$ as $\omega \to 0$ or $T \to \infty$

i.e. many oscillators at high temp
should basically spread over all levels

-or- $\boxed{\text{when } k_B T \gg \hbar\omega, \text{ discrete-ness is unimportant!}}$

MUCH on this to follow...
"high temp" means $kT \gg \hbar\omega$

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as $\omega \to 0$ or $T \to \infty$, $\hbar\omega / k_B T \to 0$

$$e^x - 1 \equiv e^{\hbar\omega/k_B T} - 1 \simeq \left(\sum_1 \frac{x^n}{n!} \right) - 1 = \left(1 + x + \frac{x^2}{2} + \ldots\right) - 1 \approx x + \frac{x^2}{2}$$

$$\Rightarrow \lim_{\substack{\omega \to 0 \\ T \to \infty}} \langle E \rangle = \lim \frac{\hbar\omega}{\hbar\omega/kT + \frac{1}{2}\left(\frac{\hbar\omega}{kT}\right)^2} = k_B T \ \checkmark$$

- high T, thermal energy so large spacing is nil.
  - recover classical result

- $\omega \to \infty$, $T \to 0$? not enough thermal E to get out of lowest level!
  $\langle E \rangle \to 0$

$\boxed{\text{Now what? calculate spectrum!}}$

how?! SKETCH for now

Can also restate our result as a probability!

i.e. $f(E) = \frac{\langle E \rangle}{\hbar\omega} = \frac{\text{avg energy}}{\text{spacing}} = \frac{1}{e^{\hbar\omega/k_BT} - 1}$

or (E per osc.)

probability that a given particle has energy E

Now: say we have our insulated, mirrored box w/ these oscillators inside, @ equil T

we want the # of particles of energy $[E, E+dE]$

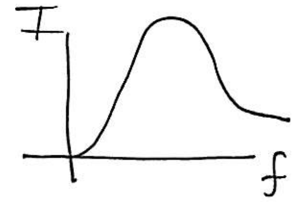
then we can find the spectrum

$$\underbrace{n(E)\,dE}_{\text{particles w/ } E \text{ between } E \to E+dE} = \underbrace{\rho(E)}_{\text{\# of states per unit volume in interval } dE} f(E)\,dE$$

(next time) $\rho(E) = \frac{8\pi}{(hc)^3} E^2$

— or — $\rho(\lambda) = \frac{8\pi}{\lambda^4}$ or $\rho(f) = \frac{8\pi f^2}{c^3}$

then $\underbrace{I(f,T)\,df}_{}$ $= \left(\dfrac{2hf^3}{c^2}\right)\dfrac{1}{e^{hf/k_BT}-1}\,df$



energy per unit surf area
per unit time
per solid angle } emitted in range $f : f+df$

-or- $I(\lambda,T) = \dfrac{2hc^2}{\lambda^5}\dfrac{1}{e^{hc/\lambda kT}-1}$ $(\lambda : \lambda + d\lambda)$

$\lambda f = c$

good: peaks ... no divergence!

as $T\uparrow$ peak shifts IR – RED – Yellow – blue

finite area (total P) at ALL T!

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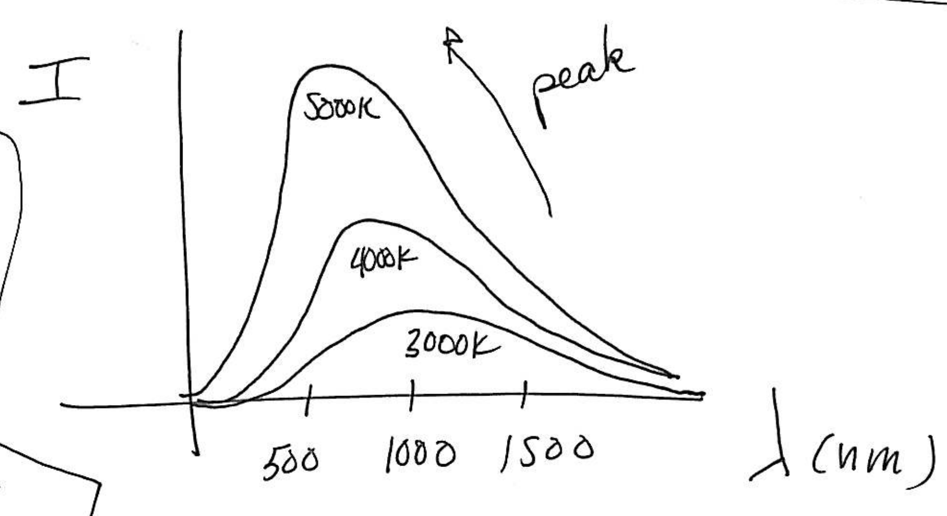
further

$\boxed{\text{Wien} \quad \lambda_{max} = \dfrac{b}{T} \quad \text{max intensity}}$

Steph-Boltzmann

$P_{tot} = \sigma T^4$



- reproduces emp. laws
- fits data, limits
- SENSE of Assumptions?

Other considerations

(1) from EM plane waves, pressure and energy density

$P = \frac{1}{3}u$ OK

(2) total power of box + body related to energy density

$P(T) = \frac{1}{4} c\,u(T)$ OK

(3) ~~Stefan~~ Stefan-Boltzmann

from Carnot cycle - power per unit area $\propto T^4$

$P(T) = \sigma T^4 \qquad \sigma = 5.67\times10^{-8}\ W/m^2K^4$ OK

(4) Wien's displacement law (Doppler)

$\lambda_{peak} = \frac{b}{T}$

further, $I(\lambda,T) = T^5 f(\lambda T)$ OK

$I \propto T^5$ and a function of (λT)

Next time : derive $I(\omega)d\omega$ – cavity radiation

derive Wein, Stephan-Boltzmann

applications

(Solar temp ; related problems)

Comparison w/ expt

OK