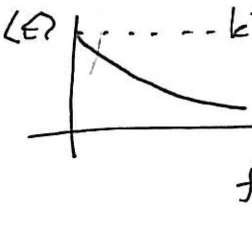
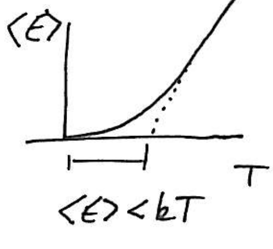
PH253
26 Jan
B Body II

Last time: accelerated charges → radiation

$$P(t) = \frac{q^2 a^2(t)}{6\pi\epsilon_0 c^3}$$

- if energy is <u>discrete</u>, i.e. only allow $\Delta E = \hbar\omega$ changes
  assumption of $\langle E \rangle = kT$ only valid at low freq!
- derived $$\langle E \rangle = \frac{\hbar\omega}{e^{\hbar\omega/kT} - 1}$$
  $\neq k_B T$!





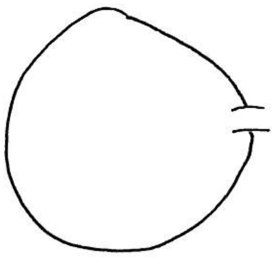
$\boxed{\text{this time}}$: forget atoms, just consider EM waves/modes.

- from last time, can't explain thermal rad by E & Malone
- <u>empirically</u>: thermal rad depends ONLY on T

  temp = measure of energy in atomic motion, avg
  <u>BUT</u> insensitive to details of its origin
  SO must our model be!

$\boxed{\text{Model}}$
- Box impenetrable to rad, perfectly mirrored
- inside, tiny bit of <u>hot gas</u> / <u>acc chgs</u> / hot matter
- wait for equil
- enough hot atoms ∴ <u>box filled w/ B Body Rad</u>

- our box has a "photon gas" inside, also a black body



- tiny hole to view ... negl. escape
  what's inside?
- consider a single oscillating charge emitting EM waves

- without container, waves escape
  - charge radiates away E ... stops eventually

- with container: perfectly reflecting walls
  - eventually every EM wave re-absorbed!
  - absorption re-accel charge, causing new emission = scattering

- Basically, charges BATHE in their own radiation
  absorb / re-emit in diff dir = scattering

  $\boxed{\text{PLUS}}$ radiation of all other atoms also absorbed & re-emitted

? what if some atoms absorb better than emit ?
hot | cold spots ?

$\boxed{\text{NO}}$ – good emitter = good absorber (Kirchhoff's Law)
– microscopic (later) – excitation & reverse have same probability
– must be so to conserve energy ... no run-away buildup

- So, in our box, emission/absorption finely balanced
  Uniform T reached, energy density uniform

from equ, require: { homogeneous
isotropic
net unpolarized
shape independent

also follows: RAD depends only on T

$\boxed{\text{? how to calculate energy density ?}}$

first, figure out EM waves inside!

EM rad inside: waves in free space! from Maxwell:

$$\nabla^2 E = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} \quad \text{(full wave eqn)}$$

one solution: real part of planewave!

$$E = A e^{i(k\cdot r - \omega t)} = E_0(r)\, e^{-i\omega t}$$

$E_0(r)$: spatial var. BC's

(where $\nabla^2 E_0 + \frac{\omega^2}{c^2} E_0 = 0$ spatial wave eqn)

reminder: k = spatial periodicity, ω = temporal periodicity $\Big\}$ $\boxed{v_{wave} = \frac{\omega}{k} = \lambda f}$

$\boxed{k = \frac{\omega}{c} = \frac{2\pi}{\lambda}}$ (vector in 3D) $\quad \underline{\omega = 2\pi f}$

$\boxed{\text{if energy is quantized, à la Planck}}$ (classical: all have avg kT)

$u = \hbar\omega$ $\quad \vec{p} = \hbar\vec{k} = \frac{\hbar\omega}{c}$ $\quad$ (cf $|\vec{p}| = \frac{E}{c}$ rel!)

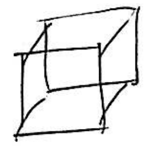
also know $\nabla\cdot E = 0 \Rightarrow \vec{k}\cdot\vec{E} = 0$

$\Rightarrow$ 2 possible E-k dirs

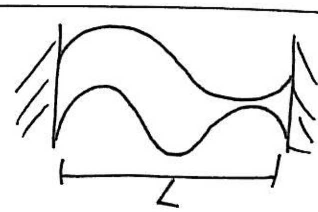
2 possible polarizations/states

(1D: $E(x,t) = E_0 \sin(\omega t)$, $E_0 = A\sin(kx)$)

- Getting the energy of all EM waves in the box? first count them!

- take, literally, a BOX 
  - equil, must have standing waves
  - other modes destructively interfere & die (transients)
  - geometry of box ⇒ allowed modes
  - any EM wave: $\frac{1}{2}$ energy in B, $\frac{1}{2}$ in E
  - use Planck to get avg energy

First: 1D model



2 mirrors (metal)

- allowed modes need $E=0$ at ends!

thus $\lambda_m = \frac{2L}{m}$ $m \in \mathbb{N}$ ($\frac{1}{2}$ wavelengths "fit")

$$\Rightarrow \boxed{\omega_m = \frac{m\pi}{L} c} \quad (\lambda f = c,\ \omega = 2\pi f)$$

allowed frequencies are discrete

$$\Rightarrow \underline{\text{field}} \quad \vec{E}_m(z,t) = A_m \underbrace{\sin\left(\frac{m\pi z}{L}\right)}_{E_0} \cos(\omega_m t)$$

(Same as $y(z,t)$ for a string ...)

Spectrum? how many modes in $[\omega, \omega+d\omega] \Rightarrow$ intensity

$$\boxed{m = \frac{L\omega}{\pi c}} \quad \text{differentiate} \quad \frac{dm}{d\omega} = \frac{L}{\pi c}$$

condition: waves fit

-or- $dm = \left(\frac{L}{\pi c}\right) d\omega$ number of modes $[\omega, \omega+d\omega]$

Classically, kT average energy per mode

($\frac{1}{2}kT$ in E, $\frac{1}{2}kT$ in B

mean energy $[\omega, \omega+d\omega]$ = (energy per mode)(modes)

$$= kT\,dm = kT\left(\frac{L}{\pi c}\right)d\omega$$

- energy of radiation is freq-indep, classically
- leads to spectrum divergence as before

Power? Standing wave = Superposn of $\longrightarrow$ $\longleftarrow$

each $\rightleftarrows$ wave takes $\Delta t = L/c$ to cross box = energy xfer

$$P = \frac{\Delta E}{\Delta t} = \left(\frac{2\times\text{energy per wave}}{\text{transit time}}\right) = \frac{kTL/\pi c}{L/c}\,d\omega$$

$\boxed{P = \frac{kT}{2\pi}\Delta\omega}$ $\Delta\omega$ = bandwidth

Can measure this: plug μwave receiver into a hot bar...

bar at 1000K, bandwidth $\Delta f = 100\,MHz$

$\Rightarrow P \sim 10^{-12}\,W$ measurable!

! how cosmic background rad discovered

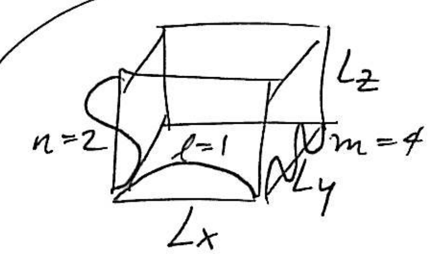
$\boxed{\text{How about a 3D box?}}$ (L_x) by (L_y) by (L_z)

- Standing waves "fit" in all 3 dirs
- need 3 integers to characterize modes!

l = # $\frac{1}{2}$ wavelengths along x

m = # along y

n = # along z



then $\boxed{k_x L_x = l\pi}$ $\quad k_y L_y = m\pi \quad k_z L_z = n\pi$

(equiv. $L = \frac{n\lambda}{2}$ in 1-D)

$$k = \sqrt{k_x^2 + k_y^2 + k_z^2}\,, \quad \omega = kc$$

$$\Rightarrow \boxed{\omega = \pi c \sqrt{\frac{l^2}{L_x^2} + \frac{m^2}{L_y^2} + \frac{n^2}{L_z^2}}}$$

frequency quant of waves in a box

? given $l, m, n \in \mathbb{N}$, how many standing wave modes ~~of~~ have freq $< \omega$

e.g., say $\underline{\omega < \pi c\sqrt{3}}$ then allowed are $\{l, m, n\} = \begin{Bmatrix} 100 \\ 010 \\ 001 \\ 110 \\ 101 \\ 011 \end{Bmatrix}$

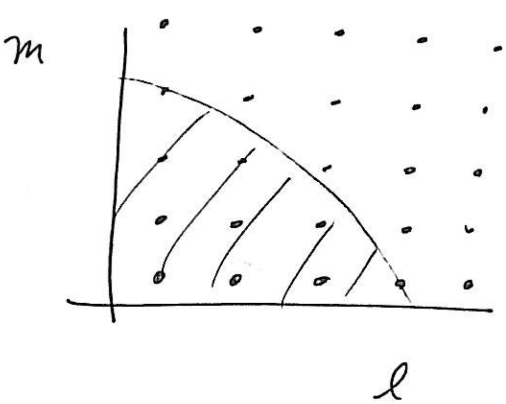
i.e. 6

Graphically: surfaces of constant ω are ellipsoids

axes $\left(\frac{L_x \omega}{\pi c}, \frac{L_y \omega}{\pi c}, \frac{L_z \omega}{\pi c} \right)$

- for a given ω, count modes inside ellipsoid
- since $l, m, n > 0$ only need one quadrant

e.g. 2D



- circle radius ω
- modes w/ lower freq inside

- if modes are closely spaced, ~~or~~ $\underline{L_i \gg \lambda}$

  \# modes $\approx$ volume

- quasi-continuous; only need SCALING to hold for an ellipsoid $V = \frac{4}{3}\pi abc$ $\Rightarrow$ $N(\omega) \approx \left(\frac{4\pi}{3}\right)\left(\frac{L_x\omega}{\pi c}\right)\left(\frac{L_y\omega}{\pi c}\right)\left(\frac{L_z\omega}{\pi c}\right)$

$$N(\omega) = \left(\frac{\omega^3}{6\pi^2 c^3}\right) V$$ V = volume of box

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Assumption OK? if $V = 1\,m^2$, $\lambda = 500\,nm$ (greenish)

$N \sim 10^{13}$! no problem

(quantum phys: neat things when this BREAKS)

Actually, forgot polarization ... 2x more

$$N(\omega) = \frac{\omega^3}{3\pi^2 c^3} V$$ modes with freq $< \omega$

in $[\omega, \omega + d\omega]$, there are

$$dN = \left(\frac{dN}{d\omega}\right) d\omega = \left(\frac{\omega^2 V}{\pi^2 c^3}\right) d\omega$$ modes

- now we just need energy per mode!

classically: $k_B T$ each

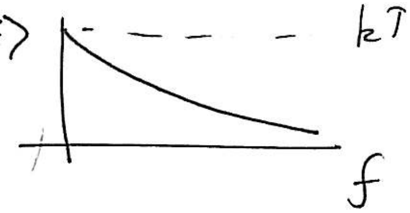
$$\Rightarrow \boxed{u(\omega)\,d\omega = \left(\frac{kT\omega^2}{\pi^2 c^3}\right) d\omega}$$ energy per volume for rad with ω in $[\omega, \omega + d\omega]$

Raleigh-Jeans ... UV catastrophe!

· now we re-use our earlier general result:

- if energy only comes in discrete bundles of $\hbar\omega$
- using Boltzmann factor for occupation

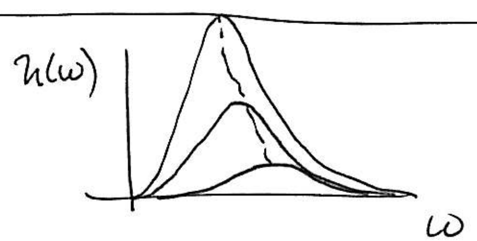
$$\langle E \rangle = (\hbar\omega)\frac{1}{e^{\hbar\omega/k_BT}-1}$$



· avg energy is k_BT only at low freq / high temp

average energy per unit volume for freq $[\omega, \omega+d\omega]$?

$$\boxed{u(\omega)\,d\omega = \frac{U(\omega)\,d\omega}{V} = \frac{dN\langle E\rangle}{V} = \frac{\hbar\omega^3}{\pi^2c^3}\left(\frac{1}{e^{\hbar\omega/kT}-1}\right)d\omega}$$



all prop. displayed by expt!

- explicit appearance of $\hbar$ means discreteness is KEY
- high ω: avg energy per mode NOT k_BT; reduced "UV cutoff" at high freq!
- Low ω: $\frac{1}{e^{\hbar\omega/kT}-1} \sim \frac{kT}{\hbar\omega} \Rightarrow u(\omega)\,d\omega \approx \frac{kT\omega^2}{\pi^2c^3}d\omega$

  reproduce classical result at low f

- peak in spectrum, shifts with T!

how about energy density versus λ? CAREFUL!

$$\lambda = \frac{2\pi c}{\omega} \Rightarrow |d\omega| = \frac{2\pi c}{\lambda^2}|d\lambda|$$

$$u(\lambda)d\lambda = \left(\frac{2\pi c}{\lambda^2}\right)u(\omega)\,d\omega$$

$\boxed{\text{keep in mind the } d\lambda,\ d\omega \text{ when "translating"}}$

$$u(\lambda)\,d\lambda = \left(\frac{8\pi hc}{\lambda^5}\right)\left(\frac{1}{e^{hc/\lambda kT}-1}\right)d\lambda$$

homework: • $u(\lambda)$ peaks at $\lambda_{peak}T = \text{const}$

Wien's displ law - scaling property

• can also show total energy $\sim T^4$

e.g. $$\frac{U(tot)}{V} = \int u(\omega)d\omega = bT^4$$

(not Stefan-Boltzmann, that's for power)

$$\boxed{P = \tfrac{1}{4}cu} \Rightarrow P = \sigma T^4 \qquad \sigma = \frac{\pi^2 k^4}{60c^2\hbar^3} \text{ by similar arg.}$$

key point: primary variable is $$\eta = \frac{\hbar\omega}{kT} = \frac{hc}{\lambda kT}$$

$$\eta = \frac{\text{photon/wave energy}}{\text{thermal energy}} = \text{only real parameter!}$$

$$u(\omega)\,d\omega = \left(\frac{\hbar}{\pi^2 c^3}\right)\left(\frac{kT}{\hbar}\right)^4 \left(\frac{\eta^3\,d\eta}{e^\eta - 1}\right)$$ "universal curve"

- max at $\eta \approx 3$
- must scale: if peak is λ_1 at T_1, λ_2 at T_2

then $\frac{\hbar\omega_1}{kT_1} = \frac{\hbar\omega_2}{kT_2}$ or $\lambda_1 T_1 = \lambda_2 T_2$

Can go further: radiation pressure $\boxed{P = \frac{1}{3}u}$

pressure ~ mean energy density "photon gas"

recall relativity: massless, $p = \frac{E}{c}$

momentum $p \Rightarrow$ force $\Rightarrow$ pressure P

$\boxed{\text{HW Hints}}$

rate of heat loss $= P_{out} - P_{in} = P_{body} - P_{surr} = \sigma(T_b^4 - T_s^4)$

balance rate of energy xport!

$\delta Q = \frac{\delta E}{\delta t} = \delta P$