Identical particles & Statistics

pt I §2 | PH253 8 Apr 2010 after variational & Bands F ①

- All particles are created equal, but some are more equal than others
  - let's go back to our original probabilistic picture and think about the consequences of particle identity
  - key to understanding practical matter arounds us!

- We started with probability <u>amplitudes</u> as a 1st <u>general principle</u>
  - just complex #s, <u>their square gives a probability</u>
  - now we know these to be wave functions' value at a point
  - go back to this, and consider more complex events

What we want is the probability that a particle from s <u>gets to</u> x

Clever notation due to Dirac for this amplitude

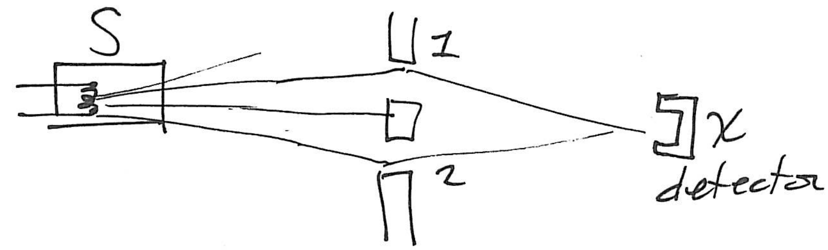
$$\langle \underbrace{\text{particle arrives at } x}_{\text{final}} \mid \underbrace{\text{particle leaves } s}_{\text{start}} \rangle$$

or even just $\langle x | s \rangle$ more compact

Still, just a <u>complex number</u>

its square gives probability (when normalized...)

earlier, 2 slit



- probability of e^- arriving at detector, both slits open?

  square of 2 amplitudes $\left\{ \begin{array}{l} P_{12} = |\varphi_1 + \varphi_2|^2 = \varphi_1^2 + \varphi_2^2 + 2\varphi_1\varphi_2 \\ \quad = |\langle x|s\rangle_1 + \langle x|s\rangle_2|^2 \end{array} \right.$

  $\Rightarrow$ interference, etc.

$\boxed{\text{general principle}}$ or 2^{nd} general principle

- when a particle can reach a given state by 2 possible routes total amplitude is the sum for the 2 routes separately

$$\langle x|s\rangle_{\text{both open}} = \langle x|s\rangle_{\text{thru }1} + \langle x|s\rangle_{\text{thru }2}$$

(before: $\varphi_{tot} = \varphi_1 + \varphi_2$ ... correspondence w/ wave function clear!)

(Could split into more possibilities, same idea)

- this gives us the probability of finding an e^- at detector x

$$P_{both,x} = |\langle x|s\rangle_1 + \langle x|s\rangle_2|^2$$ $\Rightarrow$ interference of amplitudes

how about prob. of going through 1 or 2? intermediate step?

- If we want to say about a particular route,
  total amplitude is the <u>product</u> of the amplitude to go partway with the amplitude to go the rest of the way

- so to go from S to x <u>via hole 1</u>,

$$\langle x|s\rangle_{\text{via } 1} = \langle x|1\rangle\langle 1|s\rangle$$

(read R→L time ordered)

leave 1 arrive at x

leave s arrive at 1

- so to go from $a \to b \to c$: $\langle c|b\rangle\langle b|a\rangle$

- or, now 2 slits open

$$\langle x|s\rangle_{\text{both}} = \langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle$$

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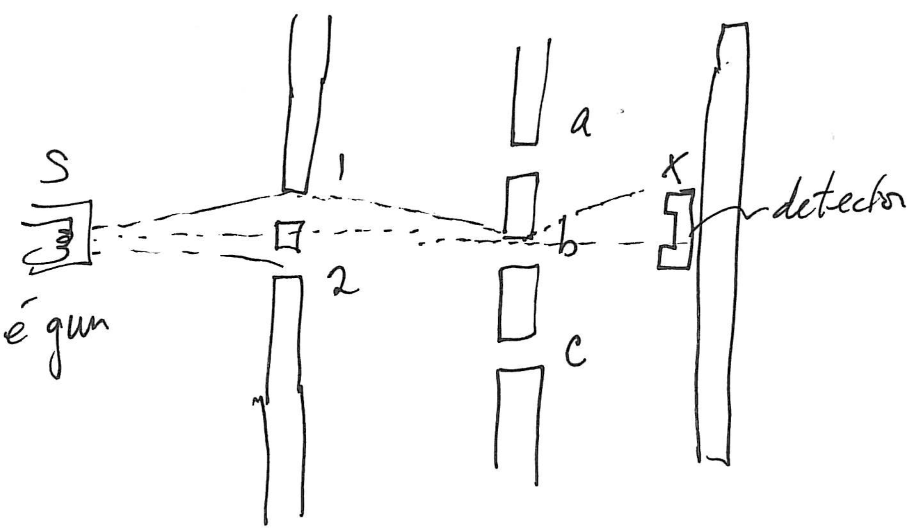
how about a complicated situation?



- 3 more holes behind
- <u>could</u> superimpose all waves...
- <u>or</u> find the amplitudes of all poss. paths from S to x

from $S \to x$ thru 1, then a, then x

$1 \to b \to x$

$1 \to c \to x$

etc <u>6 possible paths</u>

$$\langle x|S\rangle = \langle x|a\rangle\langle a|1\rangle\langle 1|S\rangle + \langle x|b\rangle\langle b|1\rangle\langle 1|S\rangle + \dots + \langle x|c\rangle\langle c|2\rangle\langle 2|S\rangle$$

$x \leftarrow a \leftarrow S$

or $$\langle x|S\rangle = \sum_{\substack{i=1,2\\ \alpha=a,b,c}}' \underbrace{\langle x|\alpha\rangle}_{\text{final}} \underbrace{\langle \alpha|i\rangle}_{\text{interm. steps}} \underbrace{\langle i|S\rangle}_{\text{start}}$$

- how does one calculate an amplitude?
  - depends on e^- <u>spin</u> or photon <u>polarization</u> in general
  - ignoring that...

Say particle goes from $\vec{r}_1$ to $\vec{r}_2$ with momentum $\vec{p}$

then $$\boxed{\langle \vec{r}_2|\vec{r}_1\rangle = \frac{e^{i\vec{p}\cdot\vec{r}_{12}/\hbar}}{|\vec{r}_{12}|}}$$ $\vec{r}_{12} = \vec{r}_2 - \vec{r}_1$ (displ)

- comes from time-dep. Schrodinger (when V indep t)

$$\Psi(x,t) = \Psi(x,0)\, e^{-iEt/\hbar}$$

- in general, amplitude depends on posn & time

- with one particle, one can think of "particle waves"
  - with more particles, dangerous... be careful
  - if particle don't interact, amplitude that one particle does something and the other something else is the product of the 2 amplitudes - independent successive processes

e.g. $\langle a|S_1\rangle$ particle 1 from $S_1 \rightarrow a$

$\langle b|S_2\rangle$ 2 from $S_2 \rightarrow b$

both things happening: $\langle a|S_1\rangle \langle b|S_2\rangle$

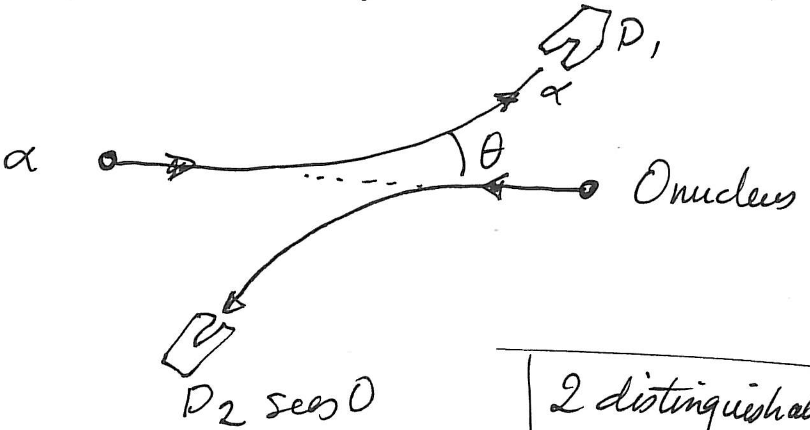
no "interference" ... like rolling 2 dice

~~one more point: say we didn't know where the particles came from before reaching slit 1 or 2~~

- In the end, we do have predictive power, just of probabilities
- probabilities for successive independent events just multiply so from an initial state we can predict the future to an extent; its just we only ever know & measure distributions & probabilities

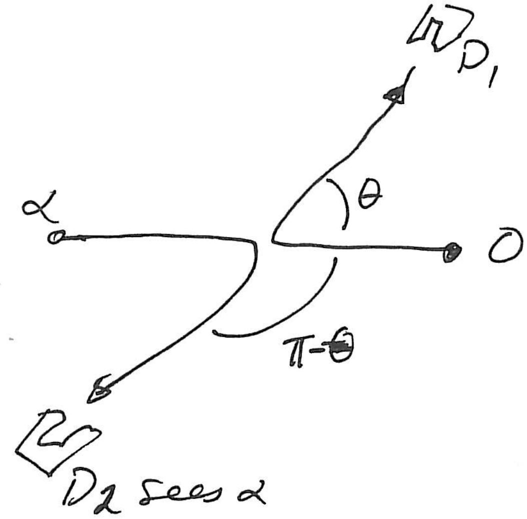
Identical particles

- what are the consequences of having indistinguishable ways for events to happen? particles?
- leads to interference of amplitudes
- illustrate by a scattering expt



vs

2 distinguishable out comes



- Shoot an α particle at an O nucleus
- from Center of mass: head-on collision (but $v_\alpha \neq v_0$ b/c $m_\alpha \neq m_0$)
- presume perfectly elastic ⇒ symmetric scattering
  - both have ⊕ charge, repel each other
  - (- C of M frame, so symmetric)
- with 2 detectors, we can measure the probability the α and O are scattered at particular angles

~~make them specific for~~

- here we can tell the α and O particles apart
- then we must <u>add probabilities</u> since the 2 are distinguishable
  - $\Rightarrow$ no interference; we can tell them apart
- let's say the amplitude of some particle hitting a detector at θ is <u>$f(\theta)$</u> then $|f(\theta)|^2$ is the probability of detection

- if we get an O nucleus at θ, then we <u>must have α at $(\pi-\theta)$</u>

- to get <u>some</u> particle at D_1, then

$$\text{Prob. \underline{some} particle at } D_1 = \underbrace{|f(\theta)|^2}_{\text{prob. } O \text{ at } D_1} + \underbrace{|f(\pi-\theta)|^2}_{\text{prob. } \alpha \text{ at } D_1}$$

(eg prob of rolling 2 or 6)

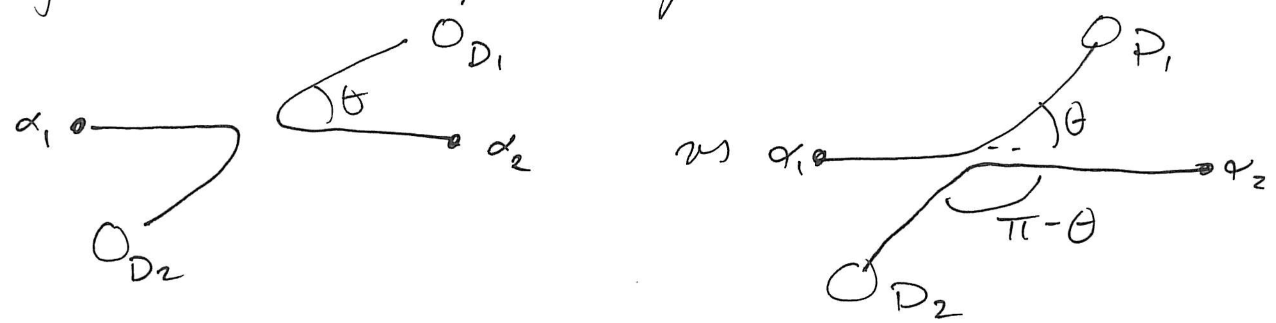
- even if our detectors <u>can't</u> tell α from O, this is the result because <u>in principle</u> we can tell them apart.

- Works fine for α particles hitting wide variety of nuclei
- <u>WRONG</u> for α hitting α!
  - in general, if the 2 particles are identical, this is NOT OK
  - why? because our adding of <u>probabilities</u> already assumed we keep track of which is which!

? What if you can't tell?

- why a big deal? for identical particles, there are 2 <u>indistinguishable</u> alternatives! $\Rightarrow$ add <u>amplitudes</u> $\Rightarrow$ interference

- try it with electrons, or 2 α particles



<u>which</u> α did D_2 detect? <u>can't say</u>! then we must <u>add amplitude</u>

$$\text{Prob } \alpha \text{ at } D_1 = \underbrace{|f(\theta) + f(\pi-\theta)|}_{}{}^2$$

f is complex $\Rightarrow$ interference <u>like 2 slit</u>

e.g. $\theta = \frac{\pi}{2}$ $\quad P = |f(\frac{\pi}{2}) + f(\frac{\pi}{2})|^2 = 4|f(\frac{\pi}{2})|^2$

for distinguishable, $P = |f(\frac{\pi}{2})|^2 + |f(\frac{\pi}{2})|^2 = 2|f(\frac{\pi}{2})|^2$!

- <u>twice</u> as much scattering at $\theta = \frac{\pi}{2}$ for α-α vs α-O!

- <u>only</u> because particles are identical!

- electrons are funnier still, thanks to spin

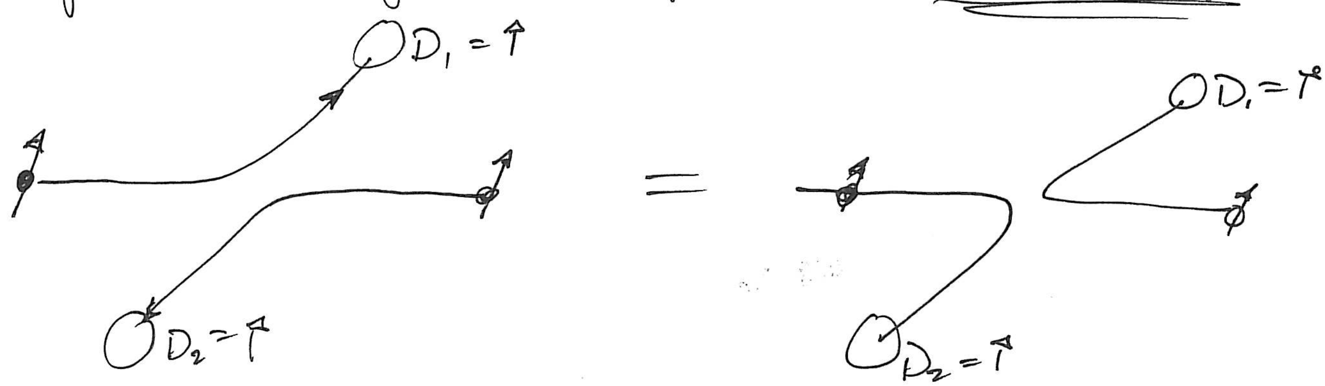
e^- have $s = \frac{1}{2}$ ... α has $s = 1$

- if spin must be considered, it is even different yet!
  - an element of distinguishability comes back

$\boxed{\text{Rule for } s = \tfrac{1}{2} \ (e^- \text{ or } p^+, \text{e.g.})}$

- if the identity of the e^- arriving at a point is <u>exchanged with another one</u>, the amplitudes interfere w/ a $\ominus$ sign!
  (i.e., opposite phase ... e^- do not look the same after 360° rot.!)

- just like interference w/ α particles, but <u>sign change</u>



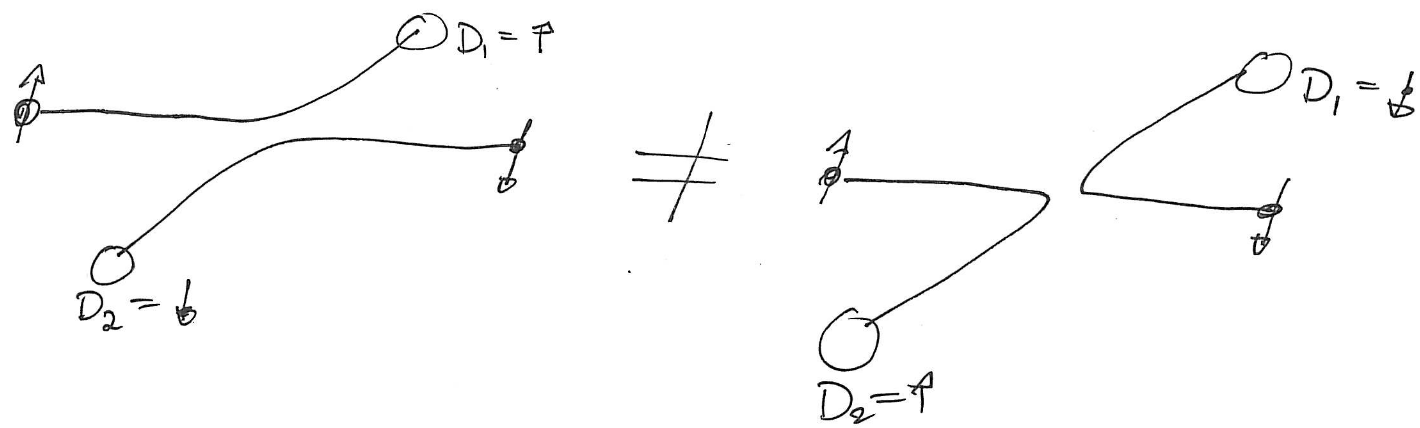
parallel spins: <u>indistinguishable</u>; can swap IDs

$$\text{prob. } e^- \text{ at } D_1 = |f(\theta) - f(\pi - \theta)|^2$$

- in this case, there is no way to tell the e^- apart
- there is interference, but w/ $\ominus$ sign

how about $\uparrow \downarrow$ collision?

that is distinguishable!



- now we <u>can</u> distinguish, <u>no</u> interference

$$\text{Prob}(e^- \text{ at } D_1) = |f(\theta)|^2 + |f(\pi-\theta)|^2$$

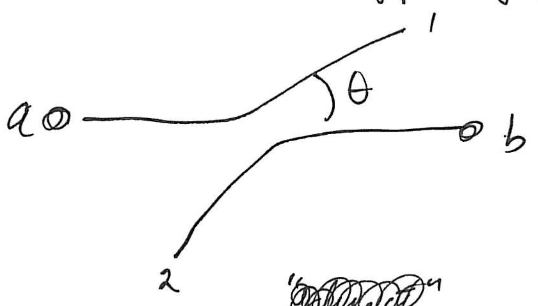
- other possibilities? can work the odds

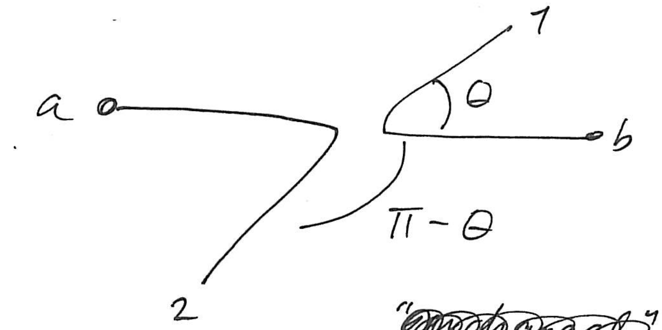
frac of cases	1	2	D_1	D_2	P
1/4	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\lvert f(\theta) - f(\pi-\theta)\rvert^2$
1/4	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\lvert f(\theta) - f(\pi-\theta)\rvert^2$
1/4	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$	$\lvert f(\theta)\rvert^2$
			$\downarrow$	$\uparrow$	$\lvert f(\pi-\theta)\rvert^2$
1/4	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\lvert f(\pi-\theta)\rvert^2$
			$\downarrow$	$\uparrow$	$\lvert f(\theta)\rvert^2$

$$P_{tot} = \frac{1}{2}|f(\theta) - f(\pi-\theta)|^2 + \frac{1}{2}|f(\theta)|^2 + \frac{1}{2}|f(\pi-\theta)|^2$$

(of course, if we use beams of e^- that are <u>not</u> unpolarized, we just need the total probability)

Overall, 2 types of particles distinguished by this behavior





Spin = integer; "Bose" particles ... like α's, or photons

ampl = (ampl direct) + (ampl exchanged)

spin = ½ integer "Fermi particles" ... e^-, p^+, neutrinos, ... n^0

ampl = (ampl direct) − (ampl exchanged)

α particle: really $2p^+$ and $2n^0$ giving net integer spin
behaves like Bose (= "Boson") ${}^4_2He$

${}^3_2He$ = $2p^+$ and $1n^0$ = Fermi-like (= "fermion")

- The reasons for this are deep, & somewhat mysterious!
- it leads to the exclusion principle

- main consequence of amplitude rules:
  - 2 Fermi particles cannot occupy the same state!
  - 2 Bose particles can!

- Fermi particles must fill up higher E levels ⇒ many particles, high E
- Bose particles can all cram into the lowest ones! e.g. photons

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Say, 2 Fermi particles collide. Prob. that $a \to 1$ and $b \to 2$?

$\langle 1|a\rangle\langle 2|b\rangle$ $a \to 1$ and $b \to 2$

Could also exchange

$\langle 2|a\rangle\langle 1|b\rangle$ $a \to 2$ and $b \to 1$

total ampl for a & b scattering?

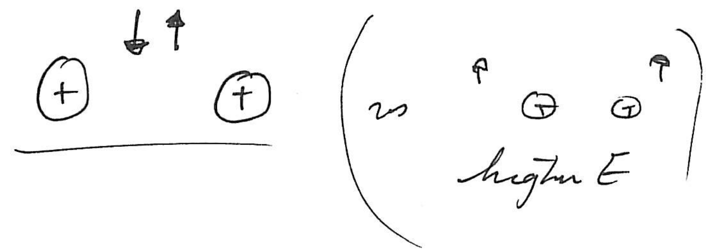
$\langle 1|a\rangle\langle 2|b\rangle - \langle 2|a\rangle\langle 1|b\rangle$

- Say spins are the same. If we want 2 in the same spot, then $1 \to 2$ and ampl $\to 0$

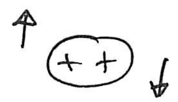
i.e., zero chance of 1 & 2 ending up in the same spot!

⇒ "like spins AVOID unlike BUNCH" $\Big\{$ if $\langle 1| = \langle 2|$ and e^- are the same, no chance to be in same place!

e.g. H_2 molecule bonding state

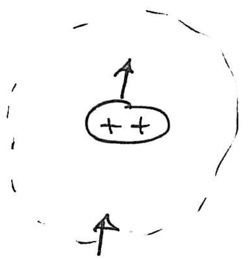


He lowest:



must be $\uparrow\downarrow$

He excited



can be $\uparrow\uparrow$ or $\uparrow\downarrow$

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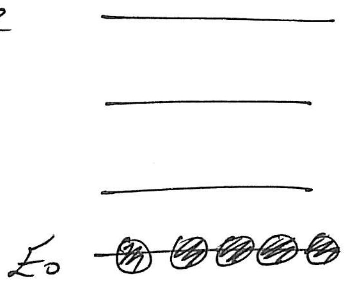
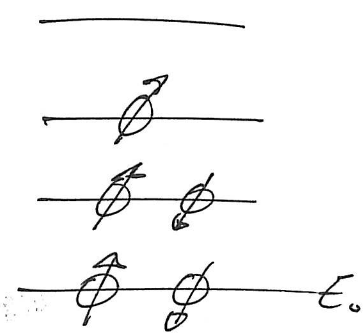
5 Bose



5 Fermi



average energy $\underline{\underline{E_0}}$

$E_{bose} < E_{fermi}$
(rest being equal)

average energy $\gg E_0$!

This has consequences for energy distribution!

<u>Classical Maxwell-Boltzmann</u> $P(E_i) \equiv f(E_i) = \dfrac{1}{Ae^{E_i/kT}}$

all ident spheres

<u>Bose-Einstein</u> $f(E_i) = \dfrac{1}{Be^{E_i/kT} - 1}$ Blackbody!

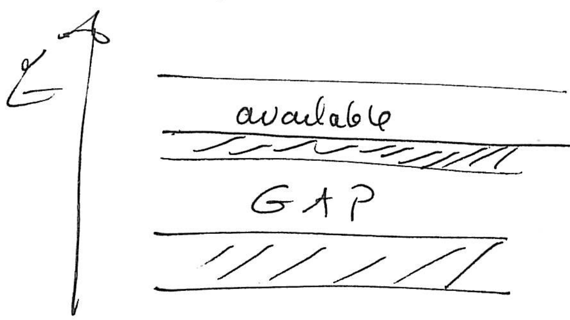
integer spin, no exclusion

Fermi-Dirac $\quad f(E_i) = \dfrac{1}{C e^{E_i/kT} + 1}$

$\frac{1}{2}$ integer spin; exclusion

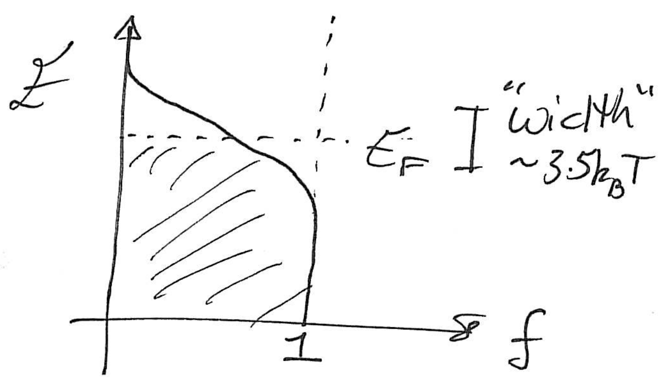
Previously: energy bands in a solid crystal

for metals, electrons fill bands and "upper" e^- have adjacent free states



top level filled $= E_F$ = "Fermi level/energy"

$$f(E_i) = \frac{1}{e^{(E_i - E_F)/kT} + 1}$$

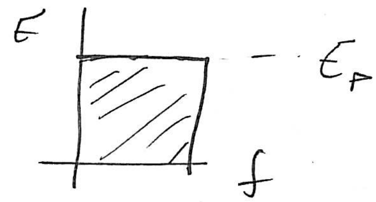


top band

note $f(E_i \ll E_F) \sim 1$

$f(E_i \gg E_F) \sim 0$

at $T=0$, basically a step function



- So, even at $T=0$, "uppermost" mobile electrons have sizable energy due to exclusion

$$E_F \sim 1.5 - 10\ eV \quad \text{metals}$$

e.g. Cu $E_F = 7.04\ eV$ so mobile upper e^- have this energy even at $T=0$!

- implies $E_F = \frac{1}{2}mv^2$ at $T=0$

$$\Rightarrow v_F = \sqrt{\frac{2E_F}{m}} \sim 10^6\ m/s\ ! \qquad (7b \text{ on HW})$$

- helps explain Ohmic conduction ... why so much scattering

- for non-interacting (free) e^- in a crystal

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \qquad N = \text{number of } e^- \quad V = \text{volume}$$

(7a on HW! "obtain" this result ...)

- can use this to get avg energy of e^-

$$E_{AVG} = \frac{1}{N}\int_0^N E_F(N')\,dN' = \frac{1}{N}\cdot\int_0^N \frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} N^{2/3}\,dN = \frac{3}{5}\frac{1}{N}\frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} N^{5/3}$$

$$= \frac{3}{5}\frac{\hbar^2}{2m}\left(\frac{3\pi^2}{V}\right)^{2/3} N^{2/3} = \frac{3}{5}E_F$$

Next time: more on Bands conduction, metals & semicond.