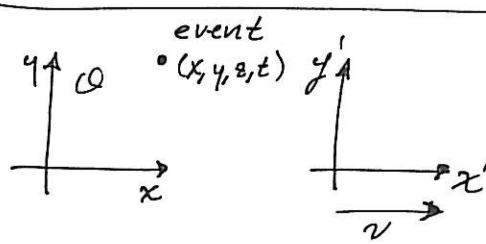
- review Lorentz & v addition; Doppler
- space-time intervals & diagrams
- momentum & energy
- homework hints



event (x,y,z,t) according to O
<u>according to O':</u>

<u>Lorentz xforms</u>

$$x' = \gamma(x - vt) \quad y' = y \quad z' = z'$$

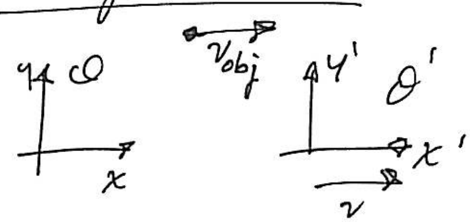
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

-or-

$$x = \gamma(x' + vt')$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

<u>velocity addition</u>



$$v_{obj} = \frac{v + v_{obj}'}{1 + \frac{v v_{obj}'}{c^2}} \qquad v_{obj}' = \frac{v_{obj} - v}{1 - \frac{v v_{obj}}{c^2}}$$

what about v_y and v_z? NO length contr... <u>but time dil!</u>

$$v_y' = \frac{dy'}{dt'} = \frac{dy'/dt}{dt'/dt} \qquad \frac{dy'}{dt} = \frac{dy}{dt} = v_y \quad \text{(no length contr in y)}$$

$$\frac{dt'}{dt} = \frac{d}{dt}\left(\gamma\left(t - \frac{vx}{c^2}\right)\right) = \gamma\left(1 - \frac{vv_x}{c^2}\right)$$

$$\Rightarrow \boxed{v_y' = \frac{v_y}{\gamma(1 - vv_x/c^2)}, \quad v_z' = \frac{v_z}{\gamma(1 - vv_x/c^2)}, \quad \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

Application: Relativistic Doppler Shift

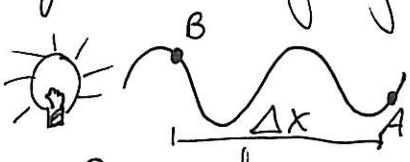
normal doppler for sound : $f = \left(\frac{v + v_r}{v - v_s}\right) f_0$

v = wave speed
v_s = source speed
v_r = receiver speed

- sound freq higher during approach ...
  ... wavefronts "pile up" so frequency is effectively larger
- ~~not OK relativistically ∴ v_r & v_s should be equiv~~
- important when source/receiver velocity ~ wave speed

? how about light? esp. traveling at v approaching c ?

moving source of light : $\boxed{\text{NO frame is special, NO ONE is in light's frame}}$



Source frame

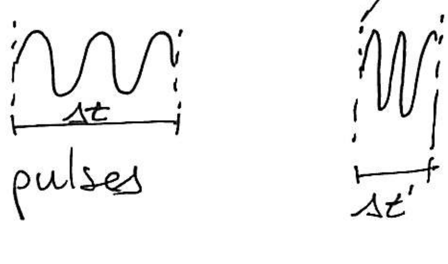
time between 2 points on light wave

$t_A - t_B = \tau \qquad \Delta x = v_w \tau$

- wave travels outward at v_w for time τ
- spacing is $\Delta x = v_w \tau$

$\boxed{\text{? what if you run \underline{towards} source at } v_0 \text{ ?}}$ Wave @ v_w

- wave crests appear to come at $v_w' = v_w + v_0$
- does NOT mean light vel. is additive !



$\Delta t = \gamma \Delta t'$ time dil/length contr squashes waves!

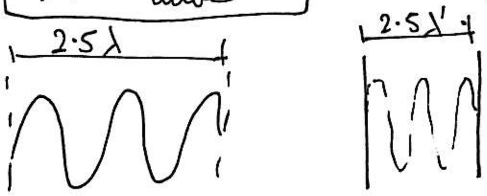
- freq/λ must change!

in bulb frame, crests come at $T = \frac{\lambda}{c} = \frac{\lambda}{v_{wave}}$

in observer frame, $T' = \gamma T$. AND crests are coming faster relative to pulse arrival

$T' = \gamma T$ and $v'_{wave} = v_{wave} + v_0$

$$\boxed{\lambda = v_{wave} T} \Longrightarrow \boxed{\lambda' = (v_{wave} + v_0)\gamma T = \frac{(v_{wave}+v_0)T}{\sqrt{1 - v_0^2/c^2}}}$$



bulb frame　　obs. frame

- with light, $v_{wave} = c$ in bulb frame
- appearance of crests NOT @ $v = c$
  info. velocity IS

$$\lambda' = \frac{(c+v_0)T}{\sqrt{1 - v_0^2/c^2}} = \frac{(c+v_0)T}{\frac{1}{c}\sqrt{c^2 - v_0^2}} = \frac{(Tc)\sqrt{c+v_0}}{\sqrt{c-v_0}}$$

note $\lambda = Tc = \frac{c}{f}$ ...

$$\boxed{\lambda' = \lambda\sqrt{\frac{c+v_0}{c-v_0}}}$$

- moving toward source @ v_0
  shortens λ toward blue

- moving away from source @ $-v_0$
  lengthens λ toward red

key in astronomy! low v: $\frac{\Delta\lambda}{\lambda} \sim \frac{v_0}{c}$

No diff if source or observer moves...

Spacetime intervals | diagrams

suppose we have 2 events A and B

A @ (x_a, y_a, z_a, t_a)

B @ (x_b, y_b, z_b, t_b)

$\Delta x = x_a - x_b$
$\Delta t = t_a - t_b$
etc

one invariant quantity all observers can agree on is the $\boxed{\text{spacetime interval}}$

$$S^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^2$$

(like dot product of 4D vectors ... $\vec{r}_A = (x, y, z, ct)$ ...
$S^2 = r_A \cdot r_B$ with convention for sign of t)

like a distance in 4D spacetime ...

$\boxed{S^2 < 0 \quad \text{or} \quad c^2 \Delta t^2 > r^2}$

"time-like" interval: enough time between events (or close enough together) that light can be present @ both events

- could be a cause-effect relationship
- NOT possible in any reference frame to be simult

$\boxed{s^2 < 0}$ cont'd

- any object w/ $v<c$ must have $s^2<0$
- will ALWAYS have $c^2\Delta t^2 > r^2$ or $\boxed{c\Delta t > r}$

real material objects travel on these "time-like" intervals

Basically, separation of events is enough that causal relationships are possible

Degree of separation characterized by proper time

$$\Delta\tau = \Delta t_p = \sqrt{\Delta t^2 - \frac{\Delta r^2}{c^2}} = \sqrt{-s^2/c^2}$$

- formally, interval measured by an observer travelling between 2 events @ const velocity
- just like notion of proper time earlier ... observer watching events unfold

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$\boxed{s^2 > 0 \text{ or } c\Delta t < \Delta r}$ "time-like interval"

- so far apart, even light can't be present @ both events
- CANNOT have causal relationships
- not meaningful to talk of PAST & FUTURE of the 2 too far away to influence!

$\boxed{s^2 > 0}$ cont'd

- can always find a frame in which events are SIMUL

separation characterized by proper distance

$$\Delta\tau = L_p = \sqrt{\Delta r^2 - c^2\Delta t^2} = \sqrt{s^2}$$

- not even light can travel on these space-like intervals
- if 2 events are separated by a ~~time~~ space-like interval
  - NO Causal relationship possible
  - their time ordering is ~~absolute~~ NOT absolute, unlike time-like

event sep.		ordering?	causality	char. by
$s^2 > 0$	spacelike	depends...	NO	$L_p = \sqrt{s^2}$
$s^2 < 0$	timelike	absolute. NOT simult.	Possible	$\tau_p = \sqrt{-s^2/c^2}$

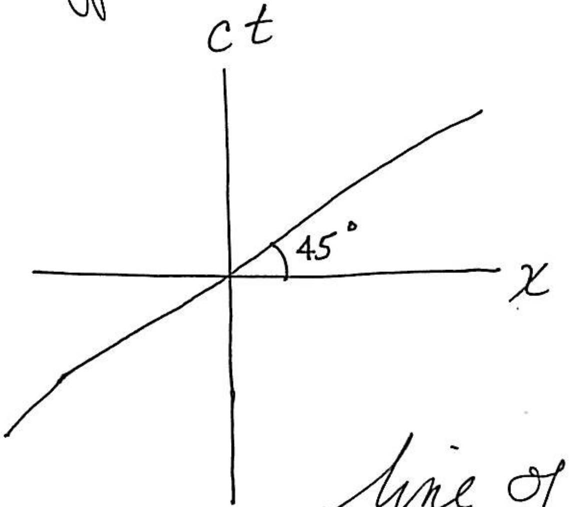
$\boxed{s^2 = 0}$ $\quad c\Delta t = \Delta r$ traveling @ $v = c$!

only light travels on these intervals

hence "light-like" intervals!

Spacetime Diagrams

- like w/ intro mechanics, position-time diagrams can help
- difference: make $v=c$ obvious, as it is special



vertical: ct

distance light travels in t

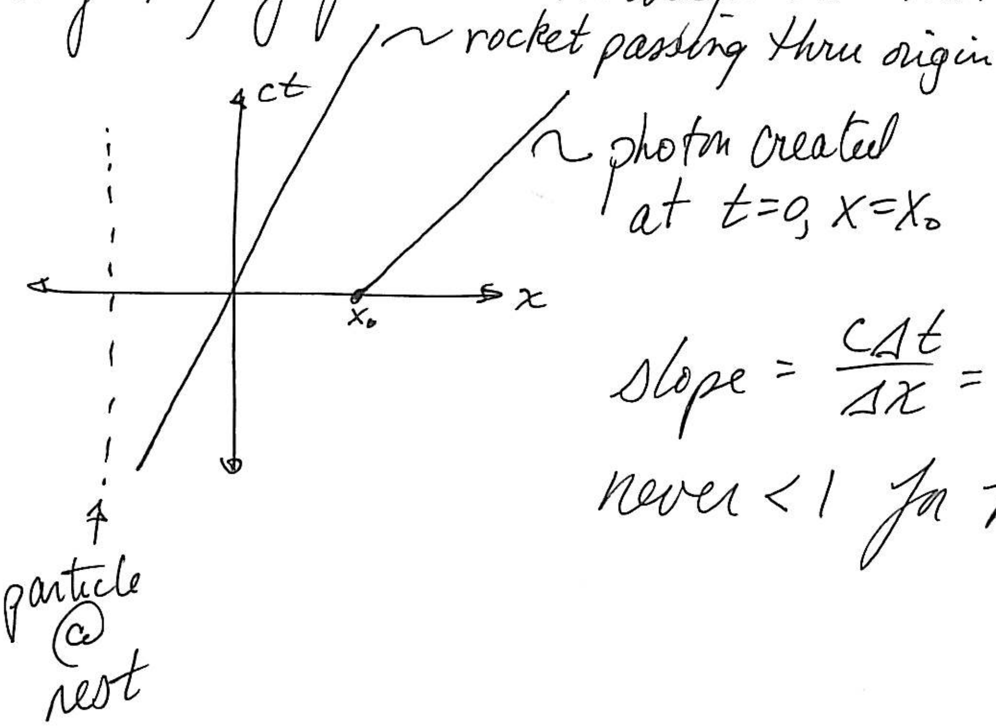
horz: distance

line of 45°: $x=ct$ or $v=c$

light!

$\boxed{\text{slope greater than } 1 \ (>45°): \ v<c}$

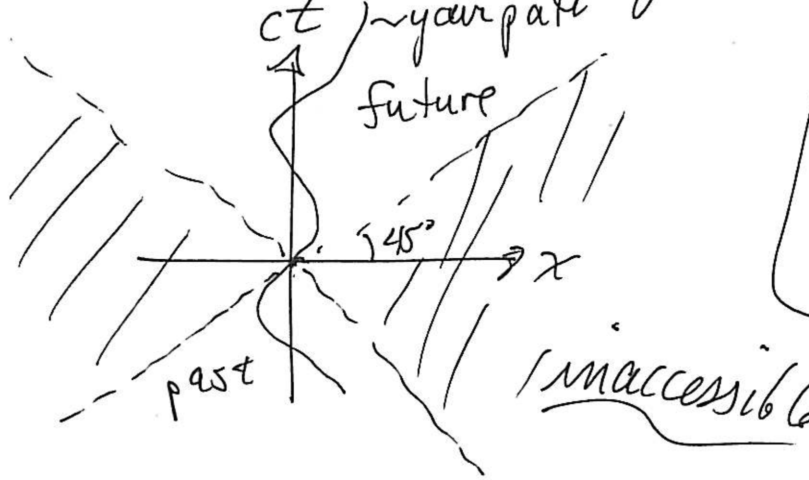
trajectory of particle through its history



$$\text{slope} = \frac{c\Delta t}{\Delta x} = \frac{c}{v}$$

never < 1 for material object

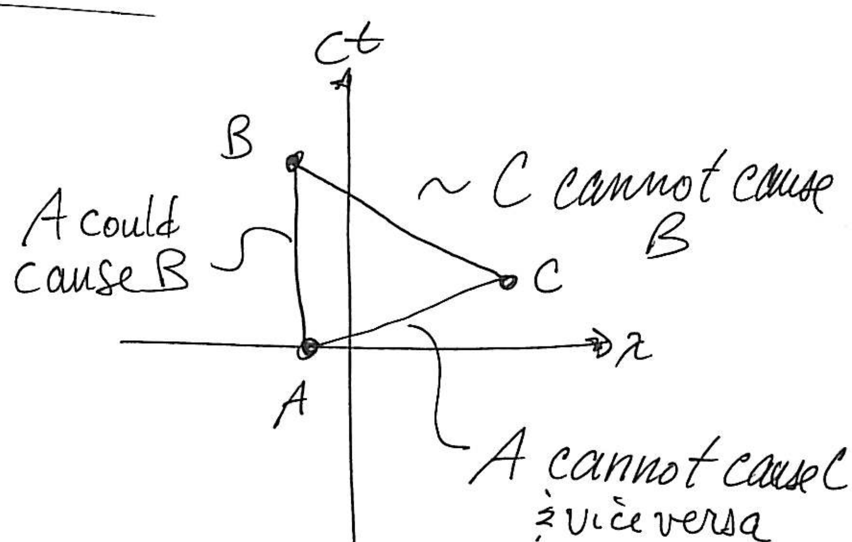
- take your position at $t=0$
  - only other regions which lie w/in $45°$ cone can possibly affect you!
  - outside your "cone", would require $v > c$ to reach / influence you!



call it a "light cone"
it defines all possible past & future influences

Can relate this to spacetime intervals!

- CONNECT 2 events



- Could be simul. in another frame

- Slope < 1: space-like, no causal
- Slope > 1: time-like, causal possible
- slope = 1: light

$\boxed{\text{※ Lorentz xforms leave } s^2 \text{ unchanged, e.g. rotations / translations on diagrams!}}$

Lots of neat math can come in here...

- group theory
- hyperbolic geometry
- transformation matricies

e.g. transformations must be linear, unitary

e.g. consider relative motion of x, x'

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = L \begin{pmatrix} x \\ t \end{pmatrix} \quad \text{with} \quad L = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma(v/c^2) & \gamma \end{pmatrix}$$

- note $L(v)L(-v) = 1$ unitary, symmetric $\left(\theta = \tanh^{-1}\left(\frac{v}{c}\right)\right)$
- actually a rotation in hyperbolic coordinates $\left(e^{\varphi} = \gamma\left(1+\frac{v}{c}\right)\right)$

  like $R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is a rotation matrix in rect. coord

- allowed xforms in SR <u>preserve</u> $S^2 = x^2 + y^2 + z^2 - c^2t^2$

  keep c <u>invariant</u>

  linear, unitary

- much more fun here... relativity $\Longleftrightarrow$ geometry

We must stick to fairly practical matters...

On to $\vec{p}$ & E

how about Momentum?

- must also maintain consv. $\vec{p}$ in relativity! all frames
- whose velocity? whose time?

for consistency, turns out the right velocity is

$$(\text{proper velocity}) = \eta = \left(\frac{\text{rate of chg of posn in observer's frame}}{\text{proper time}} \right)$$

- all agree on proper time
- velocity is still observer-dep; must be

$$\eta = \frac{dl}{dt_p} = \frac{dl/dt}{dt_p/dt} = \frac{v}{dt_p/dt}$$

from Lorentz: $\frac{dt_p}{dt} = \frac{1}{\gamma}$

$$\Rightarrow \quad \eta = \frac{v}{\sqrt{1-v^2/c^2}} = \gamma v$$

with this, we can consistently define momentum

$$\boxed{p = m\eta = \gamma m v}$$

- maintains $\vec{p}$ consv for all observers!
- e.g. watch a collision from ground vs. in one car
- w/o preferred frame, this transforms correctly

What about MASS?

Counting atoms! Modern view: invariant
only meaningful in rest frame

from p we can get energy!

e.g. energy of a body travelling at velocity v? Work from $0 \to v$

$$\Delta W = \Delta KE = \int_0^v \left(\frac{dp}{dt}\right) \cdot dl = \int_0^v \frac{dp}{dt} \cdot \frac{dl}{dt}\, dt = \int_0^v \frac{dp}{dt} \cdot v\, dt$$

chain rule: $\frac{d}{dt}(p \cdot v) = \frac{dp}{dt} \cdot v + p \cdot \frac{dv}{dt}$

$$\Rightarrow \Delta KE = \int_0^v \left[\frac{d}{dt}(p \cdot v) - p \cdot \frac{dv}{dt} \right] dt = p \cdot v \Big|_0^v - \int_0^v p\, dv$$

w/ $p = \gamma m v = \frac{mv}{\sqrt{1 - v^2/c^2}}$

$$\Delta KE = pv - \int_0^v \frac{dv\,(mv)}{\sqrt{1 - v^2/c^2}} = pv + mc^2 \sqrt{1 - v^2/c^2}\Big|_0^v$$

$$\Delta KE = pv + mc^2 \sqrt{1 - v^2/c^2} - mc^2 = \gamma m v^2 + \frac{mc^2}{\gamma} - mc^2$$

$$\Delta KE = mc^2 \left(\gamma \left(\frac{v^2}{c^2}\right) + \frac{1}{\gamma} - 1 \right)$$

USEFUL

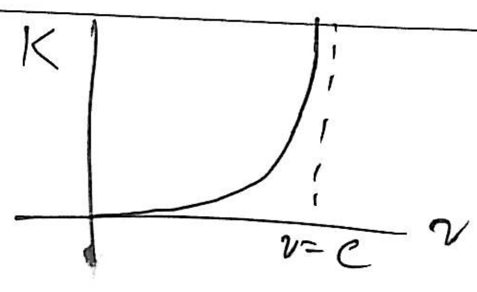
$$\frac{1}{\gamma} + \gamma\left(\frac{v^2}{c^2}\right) = \gamma\left(\frac{1}{\gamma^2} + \frac{v^2}{c^2}\right) = \gamma\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right) = \gamma$$

$$\Rightarrow \boxed{\Delta KE = (\gamma - 1) mc^2}$$

NOTE: $v \to c$, $\gamma \to \infty$
CANNOT reach $v = c$!

low v: $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \simeq 1 + v^2/2c^2$ $\quad \Delta KE \simeq \frac{1}{2} m v^2$:)

- consv of $\vec{p}$ fixed
- KE derived
- $\mathcal{E}$ consv?



if we consider E cons in all frames, we require

$$\boxed{E^2 = p^2c^2 + m^2c^4} \quad \text{with } p = \gamma m v$$

- relativistic p-E relationship

how? $p^2 = \gamma^2 m^2 v^2 = \dfrac{m^2 v^2}{1 - v^2/c^2} = \dfrac{c^2 m^2 v^2}{c^2 - v^2}$

$$p^2c^2 = \frac{c^4 m^2 v^2}{c^2 - v^2} = m^2c^4\left(\frac{v^2}{c^2 - v^2}\right)$$

$$p^2c^2 + m^2c^4 = m^2c^4\left(1 + \frac{v^2}{c^2 - v^2}\right) = m^2c^4\left(\frac{c^2}{c^2 - v^2}\right) = \underbrace{(\gamma mc^2)}_{= KE + \text{const}}{}^2 = [KE + mc^2]^2$$

$$\boxed{\gamma mc^2 = KE + mc^2 = E}$$

- relativistic total energy is $\boxed{E = \gamma mc^2}$
- identify "rest energy" or "self energy" mc^2

$$\boxed{\begin{aligned} E_{tot} &= KE + mc^2 \\ E_{rest} &= mc^2 \\ KE &= (\gamma - 1)mc^2 \end{aligned}}$$

- Bodies' energy content depends on mass!
- energy-mass equivalent (like internal E in mechanics)

Classical: $K = p^2/2m$ Rel: $E^2 = p^2c^2 + (mc^2)^2 = (\gamma mc^2)^2$

$K = (\gamma - 1)mc^2$

Some consequences:

- mass liberated as Energy – esp nuclear Rx
- if p is conserved, then so is E!

$$E^2 = (pc)^2 + (mc^2)^2$$

E p rest E

$\therefore$ $E^2 - (pc)^2 = (mc^2)^2$ implies $E^2 - (pc)^2$ is also invariant

Can also show (handy)

$$p = E\left(\frac{v}{c^2}\right) \quad \text{or} \quad c^2 p = Ev$$

$$v = \frac{cp}{\sqrt{m^2c^2 + p^2}}$$

What about massless stuff (light!)

$m = 0 \Rightarrow \boxed{E = pc}$ Compare w/ EM waves: $U = pc$!

light particles have momentum!

Further: $p = E\left(\frac{v}{c^2}\right)$ in general; $p = E/c$ massless $\Big\}$ means massless particles (ONLY) must travel at $v = c$

- Will come back to energy-momentum a lot
  - more chances to make sense of it :)

HW hints

#1 $F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v)$

if $|v| =$ const, $\gamma =$ const $\Rightarrow \frac{d\gamma}{dt} = 0$, $\frac{dv}{dt} \neq 0$

- $\frac{dp}{dt} = qvB$ circular: $\frac{dv}{dt} = \frac{v^2}{r}$ holds

#2 classic ... internet

#3 what is the distance according to the astronaut?

#4 from lecture ... $v \simeq 0.05c$

#5 again $F = \frac{dp}{dt} = \frac{d}{dt}(\gamma m v) = qE$ chain rule ...

now $\frac{d\gamma}{dt} \neq 0$; solve for dv/dt; plug in defn of γ

#6 one dir contracted, other isn't

#7 general v addition v_x & v_y

$v = \sqrt{v_x^2 + v_y^2}$ $\tan\theta = v_y/v_x$

#8 $a_x' = \frac{dv_x'}{dt'} = \frac{dv_x'/dt}{dt'/dt}$

use Lorentz ...

$v_x' = \frac{v_x - v}{1 - \frac{v_x v}{c^2}}$ $t' = \gamma\left(t - \frac{vx}{c^2}\right)$

note $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$

note v is indep of time!

#9 Let $F = F_0 \hat{x} = \frac{dp}{dt}$ ... integrate!