PH253: Compton Scattering

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Chapter 1

Compton Scattering

These notes are meant to be a supplement to your textbook, providing alternative and additional derivations of the most important Compton scattering equations. They are probably most effective after having read the relevant sections of your textbook.

1.1 Basics

Einstein, in his explanation of the photoelectric effect, modeled light as tiny massless bundles of energy – photons – which carry discrete amounts of energy based on their frequency, $E = hf = hc/\lambda$. Ignoring for the moment how we reconcile this model with wave-like behavior of light (such as diffraction or interference), from relativity we also know that the photon must have an energy

$$E = \sqrt{m^2c^4 + p^2c^2} = pc \tag{1.1}$$

In spite of the fact that photons have no mass, owing to the joint energy-momentum conservation in relativity, they must carry momentum.[i] We have the result that a photon of energy E must carry a momentum $p = E/c = hf/c = h/\lambda$.

1.2 Derivation of the Compton equation

Now, if photons are tiny particle-like bundles of energy carrying momentum, we should be able to demonstrate this fact experimentally. The most straightforward manner to demonstrate this is to scatter the photons off of another particle, such as a stationary electron. If the photon is scattered in the same fashion as a particle, a specific angular dispersion of scattering should result, and the scattered photon should lose some of its energy to the electron. The latter is particularly easy to observe in principle: if the scattered photon has a lower energy, it has a lower frequency and longer wavelength. Our classical model of radiation as electromagnetic waves would predict that

[i] That light carries momentum is also derivable from classical electromagnetic waves, but since electromagnetism is a relativistically-consistent theory this is not so surprising.

incident and scattered photons have essentially the same frequency except for very intense incident radiation. It was the observation of Compton scattering that convinced many physicists of the reality of the discrete photon model of light.

The basic idea is illustrated in Fig. 1.1 below. An incident photon of frequency f_i, energy $E_i = hf_i$, and momentum $p_i = h/\lambda_i$ strikes an electron (mass m) at rest. The photon is scattered through an angle θ, and the scattered photon has frequency f_f, energy $E_f = hf_f$, and momentum $p_f = h/\lambda_f$. As a result of the collision, the electron recoils at angle φ relative to the incident photon direction, and acquires kinetic energy KE_e and momentum p_e.

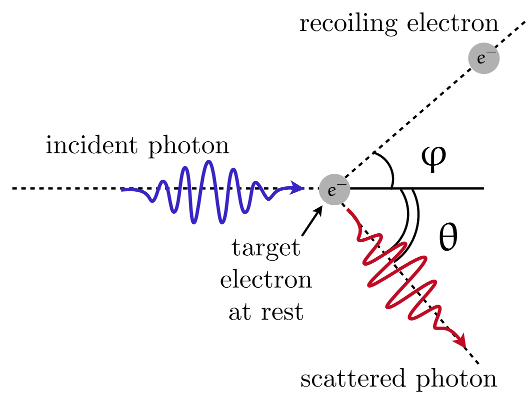

Figure 1.1: *Schematic illustration of a photon Compton scattering off of a stationary electron.*

If the photon behaves in a particle-like fashion, we can analyze this scattering process as we would any other collision: conserve energy and momentum. Conservation of energy is more straightforward. Before the collision, we have the incident photon's energy, while after the collision we have the scattered photon's energy and the electron's kinetic energy:

$$hf_i = hf_f + KE_e = hf_f + \sqrt{m^2c^4 + p_e^2c^2} - mc^2 \tag{1.2}$$

Here we have used the fact that the electron's kinetic total energy is its total energy minus its rest energy mc^2. Conservation of momentum along the horizontal and vertical directions gives, respectively,

$$p_i = p_e \cos\varphi + p_f \cos\theta \tag{1.3}$$

$$p_e \sin\varphi = p_f \sin\theta \tag{1.4}$$

In principle, the problem is now solved by suitable rearrangement of these three equations. This task is made simpler by defining dimensionless energy parameters for the incident and scattered photons and the electron, recognizing that the naturally relevant energy scale for the problem is the electron's rest energy mc^2:

$$\alpha_i = \frac{\text{incident photon energy}}{\text{electron rest energy}} = \frac{hf_i}{mc^2} \tag{1.5}$$
$$\alpha_f = \frac{\text{scattered photon energy}}{\text{electron rest energy}} = \frac{hf_f}{mc^2} \tag{1.6}$$
$$\epsilon = \frac{\text{electron kinetic energy}}{\text{electron rest energy}} = \frac{E_e}{mc^2} \tag{1.7}$$

All three quantities are dimensionless (no units) and represent the energy of each object as a fraction of the electron's rest mass. These substitutions change our energy and momentum equations to:

$$\alpha_i = \alpha_f + \sqrt{\frac{p_e^2}{m^2c^2}+1} - 1 \tag{1.8}$$
$$\alpha_i = \alpha_f \cos\theta + \left(\frac{p_e}{mc}\right)\cos\varphi \tag{1.9}$$
$$\alpha_f \sin\theta = \left(\frac{p_e}{mc}\right)\sin\varphi \tag{1.10}$$

Our experiment measures the incident and scattered photons' energy and the photon scattering angle, so the object is now to eliminate the electron's momentum p_e and scattering angle φ in favor of these quantities. We can rearrange the energy equation, square it, and solve for p_e:

$$\alpha_i - \alpha_f + 1 = \sqrt{\frac{p_e^2}{m^2c^2}+1} \tag{1.11}$$
$$\frac{p_e^2}{m^2c^2} = (\alpha_i - \alpha_f + 1)^2 - 1 \tag{1.12}$$
$$p_e^2 = m^2c^2\left(\alpha_i^2 - 2\alpha_i\alpha_f + \alpha_f^2 + 2\alpha_i - 2\alpha_f\right) \tag{1.13}$$
$$p_e^2 = m^2c^2\left((\alpha_i - \alpha_f)^2 + 2(\alpha_i - \alpha_f)\right) \tag{1.14}$$
$$p_e^2 = m^2c^2\left(\alpha_i^2 - 2\alpha_i\alpha_f + \alpha_f^2 + 2\alpha_i - 2\alpha_f\right) \tag{1.15}$$

We can now square and add the two momentum equations to eliminate φ

$$\left(\frac{p_e}{mc}\right)\cos\varphi = \alpha_i - \alpha_f\cos\theta \qquad \Longrightarrow \qquad p_e^2\cos^2\varphi = m^2c^2\left(\alpha_i - \alpha_f\cos\theta\right)^2 \tag{1.16}$$

$$\left(\frac{p_e}{mc}\right)\sin\varphi = \alpha_f\sin\theta \qquad \Longrightarrow \qquad p_e^2\sin^2\varphi = m^2c^2\alpha_f^2\sin^2\theta \tag{1.17}$$

$$p_e^2 = m^2c^2\left(\alpha_f^2\sin^2\theta + \left(\alpha_i - \alpha_f\cos\theta\right)^2\right) \tag{1.18}$$

$$p_e^2 = m^2c^2\left(\alpha_f^2\sin^2\theta + \alpha_i^2 - 2\alpha_i\alpha_f\cos\theta + \alpha_f^2\cos^2\theta\right) \tag{1.19}$$

$$p_e^2 = m^2c^2\left(\alpha_f^2 + \alpha_i^2 - 2\alpha_i\alpha_f\cos\theta\right) \tag{1.20}$$

Comparing this with our previous equation for p_e^2, 1.15, we have

$$\alpha_i^2 - 2\alpha_i\alpha_f + \alpha_f^2 + 2\alpha_i - 2\alpha_f = \alpha_f^2 + \alpha_i^2 - 2\alpha_i\alpha_f\cos\theta \tag{1.21}$$

$$-\alpha_i\alpha_f + \alpha_i - \alpha_f = -\alpha_i\alpha_f\cos\theta \tag{1.22}$$

$$\alpha_i - \alpha_f = \alpha_i\alpha_f\left(1 - \cos\theta\right) \qquad \text{or} \qquad \frac{1}{\alpha_f} - \frac{1}{\alpha_i} = 1 - \cos\theta \tag{1.23}$$

$$\tag{1.24}$$

The last equation is the famous Compton equation, which we can make more familiar by re-writing it in terms of the photons' wavelength. Noting that $\alpha = hf/mc^2 = h/\lambda mc$.

$$\frac{1}{\alpha_f} - \frac{1}{\alpha_i} = 1 - \cos\theta \tag{1.25}$$

$$\frac{\lambda_f mc}{h} - \frac{\lambda_i mc}{h} = 1 - \cos\theta \tag{1.26}$$

$$\lambda_f - \lambda_i = \Delta\lambda = \frac{h}{mc}\left(1 - \cos\theta\right) \tag{1.27}$$

$$\tag{1.28}$$

This is the more familiar textbook form of the Compton equation.

The quantity h/mc has units of length, and is known as the *Compton wavelength* $\lambda_c = h/mc \approx 2.42 \times 10^{-12}$ m. We can see that a head-on collision with the photon scattered backward at 180° gives the maximum possible change in wavelength of $2\lambda_c$. Further, the shift in wavelength $\Delta\lambda$ between scattered and incident photons is *independent* of the incident photon energy, a somewhat surprising result at first.

1.3 Electron kinetic energy

The electron's kinetic energy must be the difference between the incident and scattered photon energies:

$$\mathsf{KE}_e = hf_i - hf_f = \alpha_i mc^2 - \alpha_f mc^2 = (\alpha_i - \alpha_f)\, mc^2 \tag{1.29}$$

Solving the Compton equation for α_f, we have

$$\alpha_f = \frac{\alpha_i}{1+\alpha_i\,(1-\cos\theta)} \tag{1.30}$$

Combining these two equations,

$$\mathsf{KE}_e = (\alpha_i - \alpha_f)\, mc^2 = mc^2\left(\alpha_i - \frac{\alpha_i}{1+\alpha_i\,(1-\cos\theta)}\right) \tag{1.31}$$

$$\epsilon = \frac{\mathsf{KE}_e}{mc^2} = \frac{\alpha_i^2\,(1-\cos\theta)}{1+\alpha_i\,(1-\cos\theta)} \tag{1.32}$$

From the latter relationship, it is clear that the electron's kinetic energy can only be a fraction of the incident photon's energy, since the quantity in brackets can be at most approach, but not reach, unity. This means that there will *always* be some energy left over for a scattered photon. Put another way, it means that *a stationary, free electron cannot absorb a photon*! Scattering must occur, absorption can only occur if the electron is bound to, e.g., a nucleus which can take away a bit of the net momentum and energy.

Another important point is that while the Compton shift in wavelength $\Delta\lambda$ is independent of the incident photon energy $E_i = hf_i$, the Compton shift in photon *energy* is not. The change in photon energy is is just the energy acquired by the electron calculated above, which is strongly dependent on the incident photon energy. Further, it is apparent that the relevant energy scale is set by the ratio of the incident photon energy to the rest energy of the electron α_i. If this ratio is large, the fractional shift in energy is large, and if this ratio is small, the fractional shift in energy becomes negligible. Only when the incident photon energy is an appreciable fraction of the electron's rest energy is Compton scattering significant. Given $mc^2 \approx 511\,\mathrm{keV}$, relatively hard X-rays or gamma rays must be used to observe significant Compton scattering. Figure 1.2 shows the fractional energy as a function of incident photon energy.

What is the maximum electron energy or photon energy shift, given a particular incident photon energy? One could simply assert the maximum is clearly when $\cos\theta = -1$, i.e., $\theta = \pi$, but this is unsatisfying and perhaps a touch arrogant. We can set $d\epsilon/d\theta = 0$ to be sure:

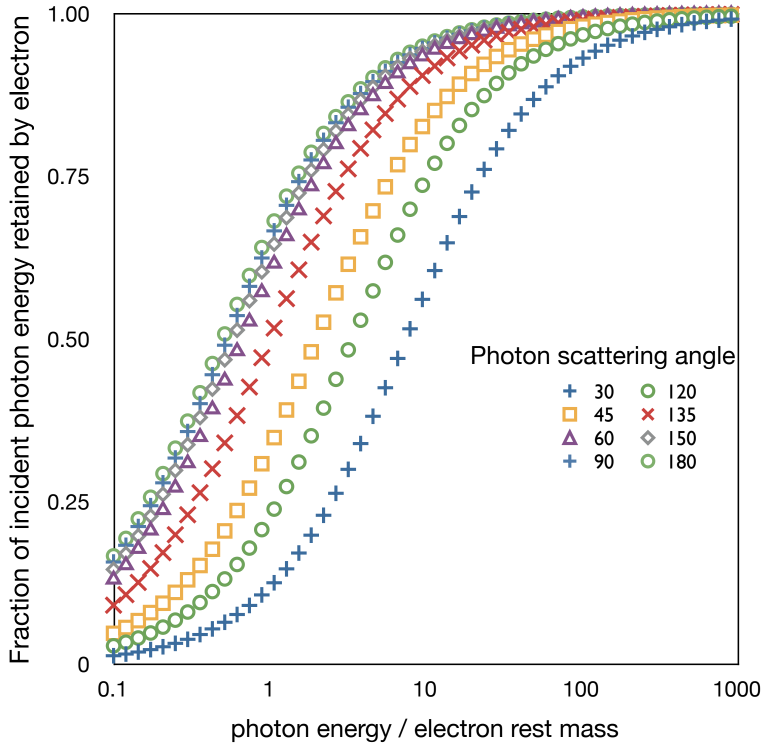

Figure 1.2: *Fraction of the incident photon energy retained by the electron as a function of incident photon energy for various photon scattering angles.*

$$\frac{d\epsilon}{d\theta} = \alpha_i^2 \left[\frac{-\alpha_i \sin\theta}{\left(1+\alpha_i\left(1-\cos\theta\right)\right)^2} + \frac{\sin\theta}{1+\alpha_i\left(1-\cos\theta\right)} + \frac{\alpha_i \sin\theta\cos\theta}{\left(1+\alpha_i\left(1-\cos\theta\right)\right)^2} \right] = 0 \tag{1.33}$$

$$0 = \sin\theta\left[-\alpha_i + 1 + \alpha_i\left(1-\cos\theta\right) + \alpha_i\cos\theta\right] \tag{1.34}$$

$$0 = \sin\theta \tag{1.35}$$

$$\theta = \{0, \pi\} \tag{1.36}$$

The solution $\theta = 0$ can be discarded, since this corresponds to the photon going right through the electron, an unphysical result. One should also perform the second derivative test to ensure we have found a maximum, but it is tedious and can be verified by a quick plot of $\epsilon(\theta)$. At $\theta = \pi$, the maximum energy of the electron thus takes a nicely simple form:

$$\mathsf{KE}_{\text{max}} = hf_i \left(\frac{2\alpha_i}{1+2\alpha_i}\right) \tag{1.37}$$

$$\epsilon = \alpha_i \left(\frac{2\alpha_i}{1+2\alpha_i}\right) = \frac{2\alpha_i^2}{1+2\alpha_i} \tag{1.38}$$

Again, we see that the maximum electron kinetic energy is at most a fraction of the incident photon energy, so absorption cannot occur for free electrons.

1.4 Other relationships

What if we want the electron's recoil angle, but don't care about the scattered photon energy? No problem, we can derive plenty of other interesting relationships. Let's go back to the momentum equations:

$$\alpha_i - \alpha_f \cos\theta = \left(\frac{p_e}{mc}\right)\cos\varphi \tag{1.39}$$

$$\alpha_f \sin\theta = \left(\frac{p_e}{mc}\right)\sin\varphi \tag{1.40}$$

Dividing them, we have

$$\tan\varphi = \frac{\alpha_f \sin\theta}{\alpha_i - \alpha_f\cos\theta} = \frac{\sin\theta}{\frac{\alpha_i}{\alpha_f} - \cos\theta} \tag{1.41}$$

We can use the Compton equation to substitute for α_i/α_f in terms of α_i alone:

$$\tan\varphi = \frac{\sin\theta}{\frac{\alpha_i}{\alpha_f} - \cos\theta} = \frac{\sin\theta}{1+\alpha_i\left(1-\cos\theta\right)-\cos\theta} = \frac{\sin\theta}{\left(1+\alpha_i\right)-\left(1+\alpha_i\right)\cos\theta} \tag{1.42}$$

$$\tan\varphi = \frac{1}{1+\alpha_i}\frac{\sin\theta}{1-\cos\theta} \tag{1.43}$$

With the aid of a rather obscure trigonometric identity, we can simplify this further. Noting

$$\frac{1-\cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right) \tag{1.44}$$

we have

$$\left(1+\alpha_i\right)\tan\varphi = \frac{1}{\tan\left(\theta/2\right)} \qquad \text{or} \qquad \frac{1}{\tan\left(\theta/2\right)} = \left(1+\frac{hf_i}{m_e c^2}\right)\tan\varphi \tag{1.45}$$

With sufficient interest, one can go on to show two other interesting relationships:

$$E_e = mc^2\left[\frac{2\alpha_i^2}{1+2\alpha_i+\left(1+\alpha_i\right)^2\tan^2\varphi}\right] \tag{1.46}$$

$$\cos\theta = 1 - \frac{2}{\left(1+\alpha_i\right)^2\tan^2\varphi+1} \tag{1.47}$$

$$\tag{1.48}$$

The first relationship is proven in the problems below.

1.5 Problems

Some of these results are already derived above. The problems below develop the some of the same relationships in a somewhat different way, however.

1. In Compton scattering what is the kinetic energy of the electron scattered at an angle φ with respect to the incident photon?

Solution: One way is simply to use the electron's energy derived in the notes and the result above. In principle, that is it: one has the energy in terms of θ, and a way to get θ from φ, so the energy can be determined from a knowledge of α_i and φ alone. This is acceptable, but inelegant. Finding a direct relationship between energy, α_i, and φ would be much nicer.

Start with the electron energy derived in the notes, with $\epsilon = E_e/mc^2$:

$$\epsilon = \frac{\alpha_i^2 (1-\cos\theta)}{1+\alpha_i(1-\cos\theta)} \tag{1.49}$$

We may use the trigonometric identity $(1-\cos\theta) = 2\sin^2\left(\frac{\theta}{2}\right)$:

$$\epsilon = \frac{\alpha_i^2 \left(2\sin^2\left(\frac{\theta}{2}\right)\right)}{1+\alpha_i\left(2\sin^2\left(\frac{\theta}{2}\right)\right)} \tag{1.50}$$

With one more identity, we can put this in terms of $\tan\left(\frac{\theta}{2}\right)$, at which point we can use the result of Sec. 1.4. The next identity is:

$$\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta} \tag{1.51}$$

which yields

$$\epsilon = \frac{2\alpha_i^2\left(\dfrac{\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)}{1+2\alpha_i\left(\dfrac{\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)}\right)} = \frac{2\alpha_i^2\tan^2\left(\frac{\theta}{2}\right)}{1+\tan^2\left(\frac{\theta}{2}\right)+2\alpha_i\tan^2\left(\frac{\theta}{2}\right)} = \frac{2\alpha_i^2}{\dfrac{1}{\tan^2\left(\frac{\theta}{2}\right)}+1+2\alpha_i} \tag{1.52}$$

Section 1.4 gives us

$$\frac{1}{\tan(\theta/2)} = (1+\alpha_i)\tan\varphi \tag{1.53}$$

Using this identity, we have the electron energy in terms of φ and α_i alone:

$$\epsilon = \frac{2\alpha_i^2}{1 + 2\alpha_i + (1 + \alpha_i)^2 \tan^2 \varphi} \tag{1.54}$$

or

$$E_e = mc^2 \left(\frac{2\alpha_i^2}{1 + 2\alpha_i + (1 + \alpha_i)^2 \tan^2 \varphi} \right) \tag{1.55}$$

2. *Park 1.2* Show that it is impossible for a photon striking a free electron to be absorbed and not scattered.

Solution: All we really need to do is conserve energy and momentum for photon absorption by a stationary, free electron and show that something impossible is implied. Before the collision, we have a photon of energy hf and momentum h/λ and an electron with rest energy mc^2. Afterward, we have an electron of energy $(\gamma - 1) + mc^2 = \sqrt{p^2c^2 + m^2c^4}$ (i.e., the afterward the electron has acquired kinetic energy, but retains its rest energy) and momentum $p_e = \gamma mv$. Momentum conservation dictates that the absorbed photon's entire momentum be transferred to the electron, which means it must continue along the same line that the incident photon traveled. This makes the problem one dimensional, which is nice.

Enforcing conservation of energy and momentum, we have:

$$\text{(initial)} = \text{(final)} \tag{1.56}$$

$$hf + mc^2 = \sqrt{p^2c^2 + m^2c^4} \qquad \text{energy conservation variant 1} \tag{1.57}$$

$$hf + mc^2 = (\gamma - 1)\, mc^2 \qquad \text{energy conservation variant 2} \tag{1.58}$$

$$\frac{h}{\lambda} = p_e = \gamma mv \qquad \text{momentum conservation} \tag{1.59}$$

From this point on, we can approach the problem in two ways, using either expression for the electron's energy. We'll do both, just to give you the idea. First, we use conservation of momentum to put the electron momentum in terms of the photon frequency:

$$\frac{h}{\lambda} = p_e \qquad \Longrightarrow \qquad \frac{hc}{\lambda} = hf = p_e c \tag{1.60}$$

Now substitute that in the first energy conservation equation to eliminate p_e, square both sides, and collect terms:

$$\left(hf + mc^2\right)^2 = \left(\sqrt{p^2c^2 + m^2c^4}\right)^2 = \left(\sqrt{h^2f^2 + m^2c^4}\right)^2 \tag{1.61}$$

$$h^2f^2 + 2hfmc^2 + m^2c^4 = h^2f^2 + m^2c^4 \tag{1.62}$$

$$2hfmc^2 = 0 \qquad \Longrightarrow \qquad f = 0 \qquad \Longrightarrow \qquad p_e = v = 0 \tag{1.63}$$

Thus, we conclude that the only way a photon can be absorbed by the stationary electron is if its frequency is zero, *i.e., if there is no photon to begin with!* Clearly, this is silly.

We can also use the second variant of the conservation of energy equation along with momentum conservation to come to an equally ridiculous conclusion:

$$hf = \frac{hc}{\lambda} = (\gamma - 1)\, mc^2 \qquad \text{energy conservation variant 2} \tag{1.64}$$

$$\frac{h}{\lambda} = \gamma m v \qquad \text{or} \qquad \frac{hc}{\lambda} = \gamma m v c \qquad \text{momentum conservation} \tag{1.65}$$

$$\Longrightarrow \qquad \gamma m v c = (\gamma - 1)\, mc^2 \tag{1.66}$$

$$(\gamma - 1)\, c = \gamma v \tag{1.67}$$

$$\frac{\gamma - 1}{\gamma} = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \qquad \text{(definition of } \gamma\text{)} \tag{1.68}$$

$$\left(\frac{\gamma - 1}{\gamma}\right)^2 = 1 - \frac{1}{\gamma^2} \tag{1.69}$$

$$\gamma^2 - 2\gamma + 1 = \gamma^2 - 1 \tag{1.70}$$

$$\gamma = 1 \qquad \Longrightarrow \qquad v = 0 \tag{1.71}$$

Again, we find an electron recoil velocity of zero, implying zero incident photon frequency, which means there is no photon in the first place! Conclusion: stationary electrons *cannot* absorb photons, but they can Compton scatter them.

3. *Ohanian 37.48* Suppose that a photon is "Compton scattered" from a proton instead of an electron. What is the maximum wavelength shift in this case?

Solution: The only difference from "normal" Compton scattering is that the proton is heavier. We simply replace the electron mass in the Compton wavelength shift equation with the proton mass, and note that the maximum shift is at $\theta = \pi$:

$$\Delta\lambda_{\text{max}} = \frac{h}{m_p c} \approx 2.64 \times 10^{-15}\,\text{m} = 2.64\,\text{fm} \tag{1.72}$$

Fantastically small. This is roughly the size attributed to a small atomic nucleus, since the Compton wavelength sets the scale above which the nucleus can be localized in a particle-like sense.

4. The Compton shift in wavelength $\Delta\lambda$ is independent of the incident photon energy $E_i = hf_i$. However, the Compton shift in *energy*, $\Delta E = E_f$–E_i is strongly dependent on E_i. Find the expression for ΔE. Compute the fractional shift in energy for a 10 keV photon and a 10 MeV photon, assuming a scattering angle of 90°.

Solution: The energy shift is easily found from the Compton formula with the substitution $\lambda = hc/E$:

$$\lambda_f - \lambda_i = \frac{hc}{E_f} - \frac{hc}{E_i} = \frac{h}{mc}\left(1 - \cos\theta\right) \tag{1.73}$$

$$\frac{cE_i - cE_f}{E_iE_f} = \frac{1 - \cos\theta}{mc} \tag{1.74}$$

$$\Delta E = E_i - E_f = \left(\frac{E_iE_f}{mc^2}\right)\left(1 - \cos\theta\right) \tag{1.75}$$

$$\frac{\Delta E}{E_i} = \left(\frac{E_f}{mc^2}\right)\left(1 - \cos\theta\right) \tag{1.76}$$

Thus, the fractional energy shift is governed by the photon energy relative to the electron's rest mass, as we might expect. In principle, this is enough: one can plug in the numbers given for E_i and θ, solve for E_f, and then calculate $\Delta E/E_i$ as requested. This is, however, inelegant. One should really solve for the fractional energy change symbolically, being both more elegant and enlightening in the end. Start from Eq. 1.76 isolate E_f:

$$\frac{E_i - E_f}{E_i} = 1 - \frac{E_f}{E_i} = \frac{E_f}{mc^2}\left(1 - \cos\theta\right) \tag{1.77}$$

$$1 = E_f\left[\frac{1}{E_i} + \frac{1}{mc^2}\left(1 - \cos\theta\right)\right] \tag{1.78}$$

$$E_f = \frac{1}{1/E_i + \left(1 - \cos\theta\right)/mc^2} = \frac{mc^2E_i}{mc^2 + E_i\left(1 - \cos\theta\right)} \tag{1.79}$$

Now plug that back into the expression for ΔE we arrived at earlier, Eq. 1.76:

$$\frac{\Delta E}{E_i} = \left(\frac{1}{mc^2}\right)\left(\frac{mc^2E_i}{mc^2 + E_i\left(1 - \cos\theta\right)}\right)\left(1 - \cos\theta\right) \tag{1.80}$$

$$\frac{\Delta E}{E_i} = \frac{E_i\left(1 - \cos\theta\right)}{mc^2 + E_i\left(1 - \cos\theta\right)} = \frac{\left(\frac{E_i}{mc^2}\right)\left(1 - \cos\theta\right)}{1 + \left(\frac{E_i}{mc^2}\right)\left(1 - \cos\theta\right)} \tag{1.81}$$

This is even more clear (hopefully): Compton scattering is strongly energy-dependent, and the relevant energy scale is set by the ratio of the incident photon energy to the rest energy of the electron, E_i/mc^2. If this ratio is large, the fractional shift in energy is large, and if this ratio is small, the fractional shift in energy becomes negligible. Only when the incident photon energy is an appreciable fraction of the electron's rest energy is Compton scattering significant. The numerical values required can be found most easily by noting that the electron's rest energy is $mc^2 = 511\,\text{keV}$, which means we don't need to convert the photon energy to joules. One should find:

$$\frac{\Delta E}{E_i} \approx 0.02 \qquad \text{10 keV incident photon, } \theta = 90° \tag{1.82}$$

$$\frac{\Delta E}{E_i} \approx 0.95 \qquad \text{10 MeV incident photon, } \theta = 90° \tag{1.83}$$

Consistent with our symbolic solution, for the 10 keV photon the energy shift is negligible, while for the 10 MeV photon it is extremely large. Conversely, this means that the electron acquires a much more significant kinetic energy after scattering from a 10 MeV photon compared to a 10 keV photon.

5. Show that the relation between the directions of motion of the scattered photon and the recoiling electron in Compton scattering is

$$\frac{1}{\tan(\theta/2)} = \left(1 + \frac{hf_i}{m_e c^2}\right) \tan\varphi \tag{1.84}$$

Solution: Let the electron's recoil angle be φ and the scattered (exiting) photon's angle be θ. Conservation of momentum gets us started. The initial photon momentum is h/λ_i, the final photon momentum is h/λ_f, and the electron's momentum we will simply denote p_e.

$$p_e \sin\varphi = p_f \sin\theta \tag{1.85}$$

$$p_e \cos\varphi + p_f \cos\theta = p_i \tag{1.86}$$

We can rearrange the second equation to isolate $p_e \cos\varphi$:

$$p_e \cos\varphi = p_i - p_f \cos\theta \tag{1.87}$$

Now we can divide Eq. 1.85 by Eq. 1.87 to come up with an expression for $\tan\varphi$:

$$\tan\varphi = \frac{p_f \sin\theta}{p_i - p_f\cos\theta} = \frac{\sin\theta}{p_i/p_f - \cos\theta} \tag{1.88}$$

We now need a substitution for p_i/p_f to eliminate p_f. For this, we can use the Compton equation, which we can rearrange to yield $\lambda_f/\lambda_i = p_i/p_f$ in terms of λ_i alone, noting $p = h/\lambda$.

$$\lambda_f - \lambda_i = \frac{h}{mc}(1 - \cos\theta) \tag{1.89}$$

$$\frac{\lambda_f}{\lambda_i} = \frac{p_i}{p_f} = 1 + \frac{h}{mc\lambda_i}(1-\cos\theta) = 1 + \frac{hf_i}{mc^2}(1-\cos\theta) \tag{1.90}$$

For the last line, we used the relationship $\lambda f = c$. Substituting this in Eq. 1.88, we eliminate p_i and p_f in favor of f_i alone, which we need in our final expression.

$$\tan\varphi = \frac{\sin\theta}{p_i/p_f - \cos\theta} = \frac{\sin\theta}{1 + \frac{hf_i}{mc^2}\left(1-\cos\theta\right) - \cos\theta} = \frac{\sin\theta}{\left(1+\frac{hf_i}{mc^2}\right)\left(1-\cos\theta\right)} \tag{1.91}$$

With the aid of a rather obscure trigonometric identity, we can obtain the desired result. Specifically:

$$\frac{1-\cos\theta}{\sin\theta} = \tan\left(\frac{\theta}{2}\right) \tag{1.92}$$

Using this in Eq. 1.91,

$$\left(1+\frac{hf_i}{mc^2}\right)\tan\varphi = \frac{1}{\tan\left(\theta/2\right)} \tag{1.93}$$

If we again define a dimensionless energy/momentum $\alpha_i = \frac{hf_i}{mc^2} = \frac{h}{mc\lambda_i} = \frac{p_i}{mc}$ the result is somewhat simpler, as is the Compton equation:

$$\left(1+\alpha_i\right)\tan\varphi = \frac{1}{\tan\left(\theta/2\right)} \tag{1.94}$$

$$\frac{\alpha_i}{\alpha_f} = 1 + \alpha_i\left(1-\cos\theta\right) \qquad \text{(Compton)} \tag{1.95}$$

This simplification has utility, as shown in the sections above, partly because it allows us to derive the electron energy in a more compact fashion, and partly because it makes the natural energy scale of mc^2 apparent.