Accelerating charges, radiation, and electromagnetic waves in solids

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Contents

1 Radiation by Accelerated Charges

1.1 Physical model and ingredients

Our task is no small one: we wish to figure out how accelerating charges emit radiation in general, and specifically find the spectrum of radiation emitted from a hot object. Why should hot objects emit radiation? In short, individual charges in atoms acquire random thermal energy, which causes them to oscillate, which causes them to radiate. We aim to calculate the spectrum of radiation emitted, within a simple toy model. Our procedure will go something like this:

1. Figure out the field from moving charges
2. Find the radiation emitted from accelerating charges, particularly for simple harmonic motion
3. From the power emitted by this radiation, find the radiation reaction force that must be present
4. Use this effective damping force to compute the equation of motion and energy of oscillating, radiating charges
5. Model a hot object as a collection of random oscillators excited by thermal energy
6. Realize the result is silly, and resort to Planck's hypothesis ...

It sounds like a lot, but we will really need nothing more than standard introductory electrodynamics and a good knowledge of the simple harmonic oscillator. As it turns out, we really only need to figure out what happens for a single charge in harmonic motion.

Subsequent sections marked with a * may be treated as optional. These sections derive formulas required for later sections (e.g., the power radiated by an accelerating charge) from more basic principles, and develop the background necessary a bit further. If you are willing to accept a few new formulas (e.g., power radiated by an accelerating charge) without derivation, they may be safely skipped.

1.2 Electric fields in different reference frames*

First, we must figure out the field due to charges in motion. Unlike length or time, *the amount of charge present is independent of reference frame.* That is, if one observer sees a charge q, all other observers will see the same charge q, independent of their frame of reference. With that in mind, consider the situation in Fig. 1.1 below, where we have a capacitor in reference frame O creating an electric field E due to a charge density σ on its plates of area A. In reference frame O' we have an observer traveling either parallel or perpendicular to the capacitor's electric field at velocity v. What electric field does the observer see?

In the capacitor's reference frame O, we know that the field between the plates is

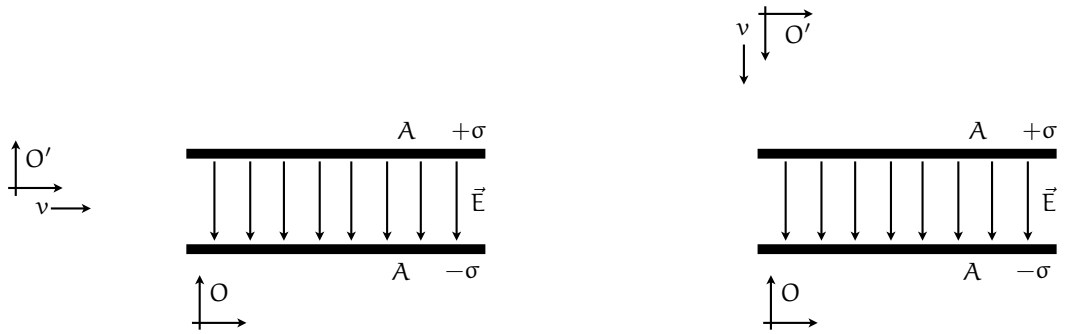

Figure 1.1: (left): *An observer in* O′ *travels at velocity* v *perpendicular to the electric field created by a capacitor in frame* O. **(right)** *An observer in* O′ *travels at velocity* v *parallel to the electric field created by a capacitor in frame* O.

$$E = \frac{\sigma}{\epsilon_o} = \frac{Q}{A\epsilon_o} \tag{1.1}$$

since the total charge on each plate Q is just σA. Consider now the case where the observer travels perpendicular to the electric field. From the observer's point of view, the dimensions of the capacitor along the direction of motion must be shortened by a factor γ. That means the area of the plates from the point of view of the observer in O′ must be smaller by a factor γ. If the total amount of charge Q is invariant, then smaller plates means *a larger apparent charge density*! Thus, the observer in O′ must see a charge density

$$\sigma' = \gamma\sigma \tag{1.2}$$

Meaning the electric field in the observer's frame must be

$$E' = \frac{\sigma'}{\epsilon_o} = \gamma\frac{\sigma}{\epsilon_o} = \gamma E \qquad \left(\vec{v} \perp \vec{E}\right) \tag{1.3}$$

The electric field for the observer moving perpendicular to the field is *enhanced* by a factor γ. Now consider the second situation, relative motion parallel to the field. In this case, the *spacing* of the capacitor is contracted according to the moving observer, but the area of the plates remains the same and thus so does the charge density. Since the field between the plates doesn't depend on the spacing,[i] but only the charge density, the field in this case is the same!

$$E' = E \qquad \left(\vec{v} \parallel \vec{E}\right) \tag{1.4}$$

In fact, there is nothing special about the field created by the capacitor, it is just like any other electric field. What we have derived, then, is the transformation of the electric field between different reference frames:

[i]The *capacitance* does depend on the spacing of the plates, but the field does not!

$$\begin{aligned} E'_{\perp} &= \gamma E_{\perp} \\ E'_{||} &= E_{||} \end{aligned} \tag{1.5}$$

Components of the electric field perpendicular to the velocity are increased by a factor γ, components parallel to the velocity are unaffected. This result holds only for charges that are stationary in one of the two frames, if charges are in motion in both frames, we will also have to consider the *magnetic* field present. Incidentally, the *force* must transform the same way, since in any frame the electric force is qE:

$$\begin{aligned} F'_{\perp} &= \gamma F_{\perp} \\ F'_{||} &= F_{||} \end{aligned} \tag{1.6}$$

Again, with the restriction that the charges in question must be at rest in at least one of the two frames. In one of the appendices to this chapter, we show how you can derive the magnetic field from the electric field of moving charges, and state the general field transformation rules when both E and B are present.

1.2.1 Field from a point charge moving at constant velocity*

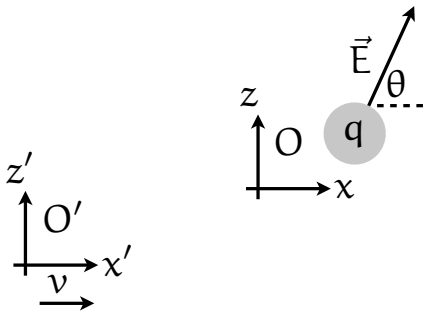

Figure 1.2: *A charge is at rest in frame* O*, while frame* O' *moves with velocity* v *and angle* θ

Armed with the rules for transforming electric fields, we can now consider what the electric field of a moving point charge looks like.[ii] We will imagine that we have a charge q traveling at velocity v along the x axis as measured by an observer in frame O', and the charge's own frame of reference will be O. Thus, in frame O the charge is at rest, while from the point of view of frame O' the charge is in motion at constant velocity v. For simplicity we will assume that the two frames have their axes aligned. Since we know that the perpendicular (z) and parallel (x) components of

[ii] In this section we follow the treatment of Purcell[**?**] closely.

$\vec{E}$ transform differently, we also know that both the magnitude and orientation of the field will be different in O'.

Let us assume that the origins of the two reference frames coincide at $t = 0$. In frame O, the charge is at rest, so the field at a distance r from the origin measured in O is:[iii]

$$E = \frac{kq}{r^2} \tag{1.7}$$

Broken down by components, we have

$$E_x = \frac{kq}{r^2}\cos\theta = \frac{kq}{x^2+z^2}\frac{x}{\sqrt{x^2+z^2}} = \frac{kqx}{(x^2+z^2)^{3/2}} \tag{1.8}$$

$$E_z = \frac{kqz}{(x^2+z^2)^{3/2}} \tag{1.9}$$

In frame O', the charge is moving at constant velocity. In order to find the field in O' we will first need to "translate" the distances as measured in O via the Lorentz transformations:

$$x = \gamma\left(x' - vt'\right) \tag{1.10}$$

$$z = z' \tag{1.11}$$

$$t = \gamma\left(t' - \frac{vx'}{c^2}\right) \tag{1.12}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1.13}$$

Based on the previous section, we also know that the component of the field parallel to the relative motion (E_x) will remain constant, but the component of the field perpendicular to the relative motion (E_z) will be augmented by a factor γ:

$$E'_x = E_x \tag{1.14}$$

$$E'_z = \gamma E_z \tag{1.15}$$

Using the field transformation and the Lorentz transformations, we can write down the field according to an observer in O' for each component:

[iii] For convenience, we use $k = 1/4\pi\epsilon_o$ in this section.

$$E'_x = E_x = \frac{kqx}{(x^2+z^2)^{3/2}} = \frac{kq\gamma\,(x'-vt')}{\left(\gamma^2\,(x'-vt')^2+z'^2\right)^{3/2}} \tag{1.16}$$

$$E'_z = \gamma E_z = \frac{kq\gamma z}{(x^2+z^2)^{3/2}} = \frac{kq\gamma z'}{\left(\gamma^2\,(x'-vt')^2+z'^2\right)^{3/2}} \tag{1.17}$$

This is something of a mess. However, our main interest here is to find the difference between the electric field observed by the moving and stationary observer at the same location (i.e., when their origins overlap). We aren't particularly worried about time dependence, issues of simultaneity, or time delays in the propagation of electromagnetic influences. Thus, we can transform the fields at time $t=t'=0$ only, which simplifies things to

$$E'_x = \frac{kq\gamma x'}{(\gamma^2 x'^2+z'^2)^{3/2}} \tag{1.18}$$

$$E'_z = \frac{kq\gamma z'}{(\gamma^2 x'^2+z'^2)^{3/2}} \tag{1.19}$$

We can already notice that the angle of the field in frame O' is

$$\tan\theta' = \frac{E'_z}{E'_x} = \frac{z'}{x'} \tag{1.20}$$

This tells us that the field in O' points along the radial direction, or that E' makes the same angle with the x' axis that the radial vector r' does. Thus, E' points radially outward from the *instantaneous position* of q. Given both components of the field in E', finding the magnitude of the field is just algebra:[iv]

$$E'^2 = E'^2_x + E'^2_z = \frac{k^2q^2\gamma^2x'^2}{(\gamma^2x'^2+z'^2)^3} + \frac{k^2q^2\gamma^2z'^2}{(\gamma^2x'^2+z'^2)^3} = k^2q^2\gamma^2\left[\frac{x'^2+z'^2}{(\gamma^2x'^2+z'^2)^3}\right] \tag{1.21}$$

$$= k^2q^2\gamma^2r'^2\left[\frac{1}{(\gamma^2x'^2+z'^2)^3}\right] = \frac{k^2q^2\gamma^2r'^2}{\gamma^6}\left[\frac{1}{(x'^2+z'^2/\gamma^2)^3}\right] \quad \left(\text{note } \frac{1}{\gamma^2} = 1-\frac{v^2}{c^2}\right) \tag{1.22}$$

$$= \frac{k^2q^2r'^2}{\gamma^4}\left[\frac{1}{(x'^2+z'^2-(v^2/c^2)\,z'^2)^3}\right] = \frac{k^2q^2r'^2}{\gamma^4}\frac{1}{(x'^2+z'^2)^3}\frac{1}{\left[1-\frac{v^2}{c^2}\frac{z'^2}{x'^2+z'^2}\right]^3} \tag{1.23}$$

$$\tag{1.24}$$

Still a mess, but we can note that $z'/\sqrt{x'^2+z'^2} = \sin\theta'$, and again use $r'^2 = x'^2+z'^2$;

[iv] Note that $r'^2 = x'^2+z'^2$.

$$E'^2 = \frac{k^2q^2}{\gamma^4 r'^4}\frac{1}{\left[1-\frac{v^2}{c^2}\sin^2\theta'\right]^3} = \frac{k^2q^2}{r'^4}\frac{\left(1-\frac{v^2}{c^2}\right)^2}{\left[1-\frac{v^2}{c^2}\sin^2\theta'\right]^3} \tag{1.25}$$

$$\Longrightarrow \qquad E' = \frac{kq}{r'^2}\frac{1-\frac{v^2}{c^2}}{\left(1-v^2/c^2\sin^2\theta'\right)^{3/2}} = \frac{q}{4\pi\epsilon_o r^2}\frac{1-v^2/c^2}{\left(1-v^2\sin^2\theta/c^2\right)^{3/2}} \tag{1.26}$$

Finally, we have the field in the frame in which the charge is moving at velocity v. What ends up happening is that the field lines end up being "squashed" along the direction of motion, so the field is much higher along the perpendicular (z') direction compared to the parallel direction (x'). Below are the field lines for a point charge moving at $0, 0.75c, 0.9c, 0.99c$ to illustrate this "relativistic compression" of field lines.

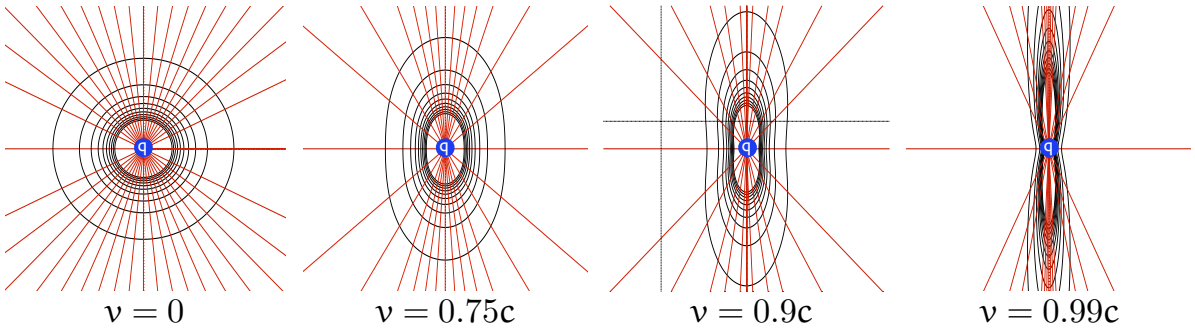

Figure 1.3: *Electric field lines (red) and contours of constant electric field (black) for a point charge moving at various velocities. At all speeds the law is an inverse square, but it is only isotropic for very low speeds.*

As the charge's relative velocity approaches c, the field becomes more and more directional. Along the horizontal axis ($z' = 0$, $\theta = 0°$), the field is reduced by a factor γ^2 compared to what it would be for a stationary charge,

$$E' = \frac{kq}{\gamma^2 r'^2} \qquad (\text{along } x') \tag{1.27}$$

while along the vertical axis ($x' = 0$, $\theta = 90°$), the field is *enhanced* by a factor γ:

$$E' = \frac{kq\gamma}{r'^2} \qquad (\text{along } z') \tag{1.28}$$

It is also the case that no static charge distribution could produce this electric field, or the electric field lines in the figure above. You can convince yourself of that by noticing that the integral of $\vec{E}\cdot d\vec{l}$ around closed paths in the figure above (say, a circle centered on the charge) is *not* zero as it must be in electrostatics. Since the line integral of $\vec{E}$ around a closed path is not zero, Maxwell's equations imply a time-varying magnetic flux. Associated with our moving charge is not just an electric field, but also a magnetic field.

1.2.2 Fields of charges that start and stop*

So far, we can figure out the fields from stationary charges, and charges in motion at constant velocity. What about charges that start or stop moving?[v] In order to find the fields in those situations, we need to remember that in free space, electromagnetic influences travel at the speed of light (you saw this in deriving the wave equation from Maxwell's equation). This "cosmic speed limit" implies the existence of electromagnetic radiation, as it turns out.

Let us imagine we have a charge q which is initially at rest, and at time $t = 0$ it is suddenly accelerated to a constant velocity v along the x axis (Fig. 1.4). We'll assume a constant acceleration a, and a duration of accelerated motion τ, where τ is very short compared to the time scale over which we observe the charge. What does the field look like surrounding the charge?

For an observer at a distance r from the origin at time T after the charge begins accelerating, it depends on whether enough time has passed for the influence of the charge's motion to travel at the speed of light over a distance r. If $r > cT$, then not enough time has passed for the "news" of the charge's motion to have reached the observer, since the news can only travel at the speed of light. Thus, for distances from the origin $r > cT$, the observer at r is unaware that the charge has now been set in motion, and the field still appears as that of a point charge! Moreover, since observers at these distances are unaware that the charge has started moving, outside a spherical shell of radius cT from the origin *the field still appears to be emanating from the charge's position at time* $t = 0$, the origin.

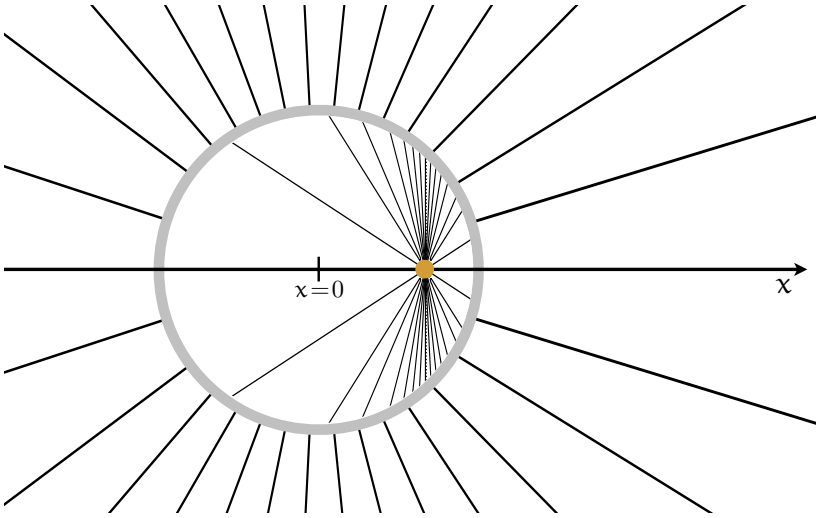

Figure 1.4: *An electron initially at rest in the lab frame is suddenly accelerated at* $t = 0$ *and moves with constant velocity* $0.8c$ *thereafter. Outside a sphere of radius* ct *from the origin, news of the charge's acceleration has not yet arrived, and the field is that of a point charge at rest. Inside the sphere, the field is that of a charge in motion at* $0.8c$*. In the grey region in between, the field lines between the two regions join.*

On the other hand, for observers within $r < c(T-\tau)$, enough time has passed that the news of the charge's acceleration has had time to reach the observer, so observers within this radius see the field of a moving point charge. Since observers at these distances are aware of the charge's motion,

[v] In this section we follow the treatment of Purcell[**?**] closely.

they also see the charge as having moved forward by an amount $x_o = v(T-\tau)+\frac{1}{2}a\tau^2$. Thus, observers inside a sphere of radius $c(T-\tau)$ see the field of a moving point charge centered at position x_o along the x axis. Figure 1.4 below illustrates the field inside and outside the "sphere of information."

As time passes, the spherical shell corresponding to the duration of the acceleration moves outward from the origin, and observers at progressively larger distances from the origin begin to see the dramatic change in the field. What happens inside the spherical shell? We know that field lines cannot cross, and that the number of field lines must remain the same so long as the amount of charge q remains constant (field lines can't stop or start in empty space). Thus, the field lines inside and outside the shell must connect to each other within the shell. These connecting lines will no longer be purely radial (either from the origin or the charge's later position), implying that within the shell the field has a *transverse* component as well. In essence, as the charge accelerates it "sheds" part of its electric field within the spherical shell, which travels outward at c. The presence of an electric field in the shell implies that energy is being carried away from the charge, what we usually call *electromagnetic radiation.* This means that the charge is losing the energy contained in the electric field within the shell, and if it is losing energy it must be experiencing a force due to the emission of radiation. We will derive this resistive force in later sections.

As another example, we could consider a charge which suddenly *stops* instead, as shown in Fig. 1.5. In this case, our charge is moving with velocity $v = 0.8c$ until reaching the origin at $t = 0$, and which point it suddenly stops. Outside a sphere of radius ct, the news of the charge's deceleration has not been received, and thus the field appears to be that of a point charge in motion at $0.8c$, emanating from a point vt past the origin on the x axis. Within the spherical shell, information of the charge's deceleration has had sufficient time to propagate, and the field appears as that of a point charge at rest at the origin.

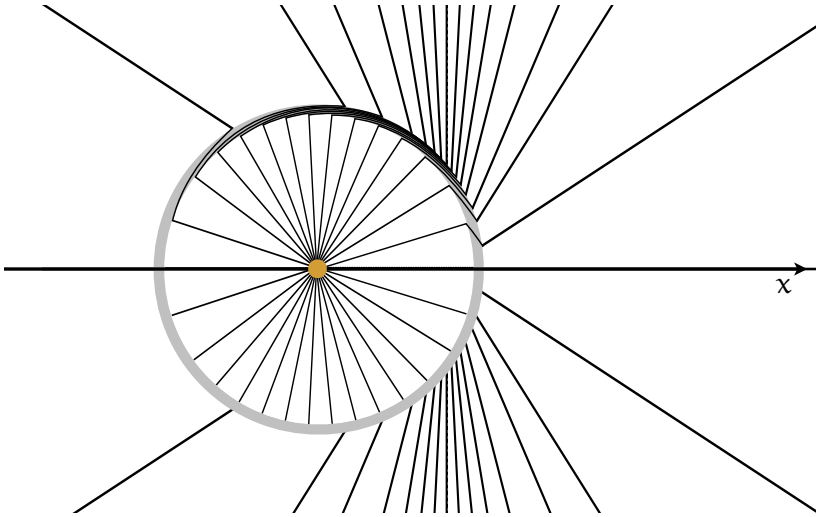

Figure 1.5: *An electron that was moving with constant velocity* $0.8c$ *reaches the origin at* $t = 0$*, suddenly stops, and remains stationary thereafter. Outside a sphere of radius* ct*, the field lines are those of a charge in motion at* $0.8c$*, while inside the sphere the field is that of a point charge at rest. In the spherical shell corresponding to the duration of the deceleration, field lines from inside and outside the sphere connect (shown for the upper portion of the figure only).*

Inside the spherical shell representing the deceleration period, we have shown how the field lines

connect in the upper half of the figure. The precise shape of the kinks depends on the details of the acceleration, and are of little interest here. What is important is that they are *transverse* with almost no radial component, and this field within the shell propagates outward as a pulse. Further, given that the electric field is a function of time, there will also be a magnetic field associated, and together the two fields make up an electromagnetic pulse. Figure 1.6 below shows contours of constant power for charge undergoing uniform acceleration along the horizontal axis. In the next section, we'll derive the formula for the radiated power.

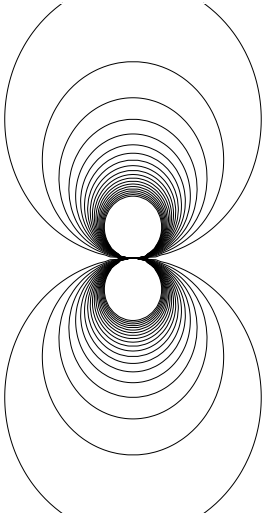

Figure 1.6: *Radiation pattern of a charge accelerating to along the horizontal axis. The different curves are contours of constant emitted power per unit area, in decreasing magnitude further from the origin.*

1.3 Radiation of accelerating charges

In your introductory physics course, you learned about the electromagnetic waves produced by an antenna, and the general fact that accelerating charges emit electromagnetic radiation. In the previous section, we established that a charge that suddenly (but smoothly) accelerates "sheds" part of its electric field as a spherical shell of radiation. The question we wish to answer now is *how much* radiation is emitted by an accelerating charge?[vi]

1.3.1 A smoothly accelerating point charge

We will now consider a charge q which has been traveling at velocity v_o along the x axis for a long time, and suddenly at time $t=0$ it decelerates smoothly for a time τ (implying acceleration $a=v_o/\tau$) until it comes to rest as shown in Fig. 1.7.[vii]

From the time the particle begins its deceleration until it stops, it will have moved a distance $x=\frac{1}{2}v_o\tau$ further along the x axis where it comes to rest. Since we presume $v_o \ll c$, this distance is tiny compared to the other relevant distances, viz., the distance traveled by light over the time

[vi] This section closely follows Appendix B of [**?**].

[vii] In this section we follow the treatment of Purcell[**?**] closely.

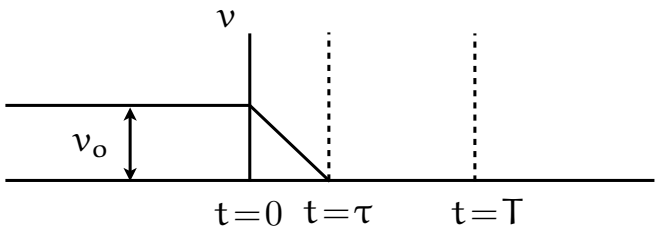

Figure 1.7: *Velocity versus time for the point charge. It travels with velocity* $v_o \ll c$ *until time* $t = 0$*, at which point it smoothly decelerates to rest over a time* $t = \tau$*.*

scales given. At a given position from the charge at some time $t = T \gg \tau$, what does the field look like? We have to be careful again to take into account the fact that the influence of the charge's motion travels outward from the charge at $v = c$, so an observer at a distance d doesn't 'get the news' that the charge stopped until a time $\delta t = d/c$ later! At a time T after the start of the deceleration, observers farther away than $R > cT$ cannot know that the charge has stopped yet, since that would imply communication faster than the speed of light. On the other hand, observers within a radius $R < c(T-\tau)$ will already see the charge as stationary. Within a thin shell of width $c\tau$ at a distance $cT < R < (T-\tau)$, observers see the charge in the midst of its deceleration.

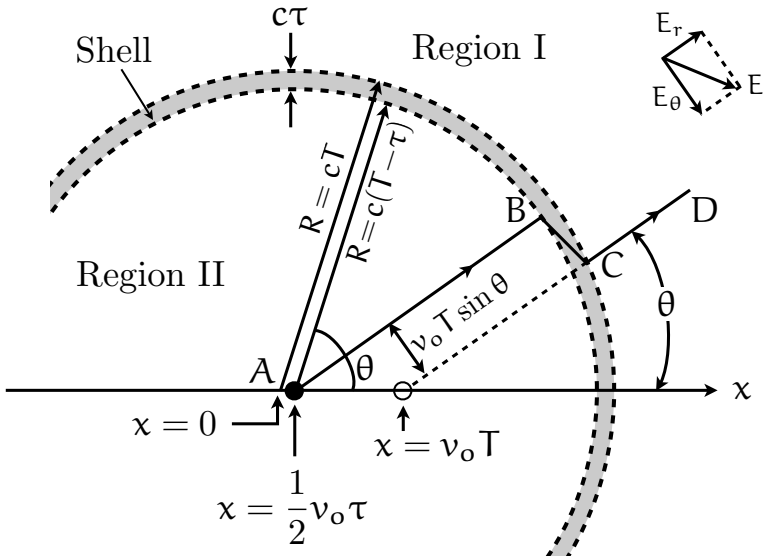

Figure 1.8: *Schematic space diagram of the instant* $t = T \gg \tau$*, a long time after the particle has decelerated. Inside a radius* $R < cT$ *(Region II), observers see a particle at rest at position* $x = \frac{1}{2}v_o\tau$*. Outside a radius* $R < c(T-\tau)$ *(Region I), observers still see a charge in motion at constant velocity* v_o *located at* $x = v_o T$*. A shell of width* $c\tau$ *between the two regions represents the transition to the field of a moving to a stationary charge. One electric field line (*ABCD*) is shown through all three regions. The diagram in the upper right shows the electric field and its component inside the shell along segment* BC*.*

We've tried to depict this situation in Fig. 1.8, there are three distinct regions of space categorized by what an observer would see:

Region I: Outside the thin shell, $R > cT$, news of the charge's deceleration has not reached observers, so *the field must still look like that of a charge moving at constant velocity* v_o*!* In fact, it must appear that nothing has changed, so the field is as though the charge is still moving at v_o, and at position $x = v_o T$ after a time T. At any given time in Region I, the field appears to emanate from the present position of the charge *as if it were still in motion*, $x = v_o T$, but compressed along the axis perpendicular to the direction of motion compared to those of a point charge. One such field line is shown as segment CD in the figure above.

Region II: In this region, time enough has passed for information about the charge's deceleration to reach observers. Inside a radius $R < c(T-\tau)$, see the charge at rest, and the field is simply that of a stationary point charge at position $x = \frac{1}{2} v_o \tau$. The field lines emanate radially from the charge's position, like segment AB in the figure.

The Shell: Between regions I and II, $cT < R < (T-\tau)$ observers are at just the right distance to see the charge in the midst of its deceleration. Our task is to find the field in this region, since we already know the field in the other two regions! Since field lines cannot start and stop in empty space, the field must be represented by segment BC.

What should the field look like in the transition region? Gauss' law provides an answer. Consider a field line like the one connecting points A and B (which would actually form a cone around the x axis). This cone contains a certain amount of the flux from the charge q. If another field line, like CD, makes the same angle with the x axis, the cone it defines must contain exactly the same amount of flux. Since field lines can never cross, it must be true that AB and CD are part of the same field line, connected by segment BC.

What of the field in the shell? It must be along the segment BC connecting the field lines AB and CD, and therefore has radial and tangential components. In region II, we have a simple stationary point charge, with a purely radial field. Gauss' law tells us that the flux through the surface defining the inner surface of the shell can only depend on the charge enclosed within, and the flux itself is determined purely by the radial portion of the field. Since we only ever have the single charge q inside region II, the *radial portion of the field cannot change when going from region II to the shell.* In region II the radial component of the field is just that of a point charge, and it must be the same inside the shell:

$$E_r = \frac{q}{4\pi\epsilon_o R^2} = \frac{q}{4\pi\epsilon_o c^2 T^2} \tag{1.29}$$

Noting from the geometry of the figure[viii]

[viii] Look at a little triangle with BC as a hypotenuse.

$$\tan\theta = \frac{E_r}{E_\theta} = \frac{c\tau}{\nu_o T \sin\theta} \tag{1.30}$$

This gives us the tangential portion of the field:

$$E_\theta = E_r \frac{\nu_o T \sin\theta}{c\tau} = \frac{q\nu_o \sin\theta}{4\pi\epsilon_o c^3 T\tau} \tag{1.31}$$

We can also remember that ν_o/τ is just the acceleration and $R = cT$, which gives us

$$E_\theta = \frac{qa\sin\theta}{4\pi\epsilon_o c^2 R} \tag{1.32}$$

One striking thing about this result is that the tangential field goes as $1/R$, not $1/R^2$! As time goes on, meaning R increases, the tangential field will eventually be much stronger than the radial one, owing to its slower $1/R$ decay. Just to review: in region II we have the field of a point charge at constant velocity, which has both radial and tangential components. In region I, we have the purely radial field of a stationary point charge. During the deceleration, the tangential component of the field is 'lost' as radiation, and this radiation emanates outward from the charge at velocity c making up a thin shell of width $c\tau$.

Our next question is then how much energy must be 'lost' by the charge during deceleration, i.e., how much energy is carried away by radiation? This amounts to finding the energy stored in the tangential field within the spherical shell, since the radiation happens only during the deceleration of the charge. The energy density (energy per unit volume) is readily calculated:

$$u_\theta = \frac{1}{2}\epsilon_o E_\theta^2 = \frac{q^2 a^2 \sin^2\theta}{32\pi^2\epsilon_o c^4 R^2} \tag{1.33}$$

The volume of the shell is just surface area times thickness, or $4\pi R^2 c\tau$, so the energy carried away in the tangential electric field shed by the charge is:

$$U_\theta = \frac{q^2 a^2 \tau \sin^2\theta}{8\pi\epsilon_o c^3} \tag{1.34}$$

Of course, in addition to the tangential electric field there must also be a magnetic field whenever charge is in motion. We know that the magnetic field carries the same amount of energy as the electric field, so we can simply double the result above:

$$U_\theta = \frac{q^2 a^2 \tau \sin^2\theta}{4\pi\epsilon_o c^3} \tag{1.35}$$

This is the energy emitted at an angle θ with respect to the x axis. More convenient is the total emitted energy, which means we should average over all θ. You can convince yourself that the average value of $\sin^2\theta$ over a sphere is $\frac{2}{3}$, giving a total energy of

$$\langle U_\theta \rangle = \frac{q^2 a^2 \tau}{6\pi\epsilon_o c^3} \tag{1.36}$$

Here the angle brackets just remind us that we are dealing with an average quantity. Note that since the shell represents the *entire* deceleration process over time τ, what we have just found is the energy dissipated by radiation during the whole deceleration process. What is striking about this result is that the dependence on R has cancelled entirely. This much energy simply emanates outward from the charge at speed c from the site of the charge's deceleration. The deceleration happens over a time τ, so we could also define a power radiated during the deceleration process:

$$P_{rad} = \frac{\langle U_\theta \rangle}{\tau} = \frac{q^2 a^2}{6\pi\epsilon_o c^3} \qquad \text{total emitted power, } E \text{ and } B \text{ fields} \tag{1.37}$$

This is the famous Larmor formula for the power radiated by an accelerating charge. If you wanted the power emitted at a particular angle, you could skip the averaging step above and find the angle-resolved power:

$$P_{rad} = \frac{q^2 a^2 \sin^2\theta}{4\pi\epsilon_o c^3} \qquad \text{emitted power from } E \text{ at angle } \theta \tag{1.38}$$

If we divide that by $4\pi R^2$, we have the power per square meter of surface area at a distance R radiated in direction θ:

$$P_{rad} = \frac{q^2 a^2 \sin^2\theta}{16\pi^2\epsilon_o c^3 R^2} \qquad \text{emitted power from } E \text{ per unit area at angle } \theta \tag{1.39}$$

What is interesting about these results is that is the *square* of acceleration that enters the equation for power, meaning the sign of the acceleration is irrelevant. Acceleration and deceleration give the same result, consistent with relativity – after all, what is deceleration in one reference frame is acceleration in another. As it turns out, P_{rad} is also independent of reference frame. Finally, and we've saved the best for last, the Larmor formula above is much more general than we have a right to expect. It works not only for instantaneous acceleration, as we derived it above, but for variable acceleration as well, such as simple harmonic motion.

1.3.2 Oscillating Charges*

As an example, let's consider a charge in simple harmonic motion,[ix] following the trajectory $x(t) = x_o \cos\omega_o t$. We know that in this case the acceleration is $a = -\omega^2 x = -\omega_o^2 x_o \cos\omega_o t$ at any instant, where $\omega_o = 2\pi f_o$ is the natural (angular) frequency of oscillation, x the instantaneous position, and x_o the amplitude of oscillation.[x] Can we just square this and plug it in the Larmor equation? We should be a bit more careful than that – plugging in this acceleration would give us the instantaneous power, but what is more useful is the average power emitted over one full cycle of oscillation. For

[ix] In this section we follow the treatment of Feynman[**?**] closely.

[x] Here we're ignoring any irrelevant phase factor, since we're only talking about a single oscillator. More generally, we should write $x(t) = x_o e^{i\omega_o t}$.

that we want the average of a^2 over one full cycle. Since the average of $\cos^2 \omega t$ is 1/2, the average squared acceleration per cycle is[xi]

$$\langle a^2 \rangle = \langle -\omega_o^4 x_o^2 \cos^2 \omega_o t \rangle = -\omega_o^4 x_o^2 \langle \cos^2 \omega_o t \rangle = \frac{1}{2}\omega_o^4 x_o^2 \tag{1.40}$$

Thus the total emitted power must be

$$P = \frac{q^2 \omega_o^4 x_o^2}{12\pi\epsilon_o c^3} \tag{1.41}$$

The fact that the oscillator is emitting power means that it is losing energy, and it must therefore be losing amplitude. Even in empty space, a freely-oscillating charge would eventually stop oscillating due to radiation losses – there is no friction, viscosity, or drag, but nevertheless dissipation occurs via radiation. Physically, our accelerating charge emits radiation – electromagnetic waves – at its resonance frequency ω_o.[xii] In fact, the charge could have begun its oscillation in the first place by being excited by incident radiation! One accelerating charge emits EM waves for a time (which we will determine below), until the radiative dissipation fritters away too much of its energy. These EM waves can be absorbed by another nearby charge, which will set *it* in oscillatory motion at the same frequency, leading to further emission of radiation, which can excite another charge ... and now we have propagation of radiation through a medium.

For a "lossy" oscillator, such as the mathematically-equivalent RLC circuit or a mass-spring-damper system, we typically calculate the quality factor Q, a measure of the rate of energy loss through viscous damping. It is defined as[xiii]

$$Q = 2\pi \frac{\text{total energy of oscillator}}{\text{rate of energy loss per radian}} = \omega_o \frac{\text{energy stored}}{\text{power loss}} = \omega_o \frac{\mathcal{E}}{d\mathcal{E}/d\theta} = \frac{\omega_o \mathcal{E}}{P} \tag{1.42}$$

Another equivalent definition of Q is $Q = \omega/\Delta\omega$, where $\Delta\omega$ is the width of the resonance curve. Using $d\mathcal{E}/dt = P$, for a given Q, the rate of energy loss (power dissipation) of the oscillator can be found in terms of Q, $\mathcal{E}$, and ω_o:

$$P = -\frac{d\mathcal{E}}{dt} = -\frac{\omega \mathcal{E}}{Q} \tag{1.43}$$

$$\implies \qquad \mathcal{E} = \mathcal{E}_o e^{-\omega_o t/Q} \tag{1.44}$$

where the initial energy of the oscillator is $\mathcal{E}_o$ at $t = 0$. The minus sign is just there to signify that energy is being lost, not gained, so $d\mathcal{E}/dt$ must be negative. The energy of the oscillating charge exponentially decays with a time constant of Q/ω_o, just as we would find for an RLC circuit.[xiv]

[xi] Since ω and x_o are constants, we can bring them out of the averaging brackets.

[xii] There will be a spread in the emitted frequencies dictated by the degree of dissipation and the "quality factor" of the oscillator, which we discuss below.

[xiii] In this section we will use $\mathcal{E}$ for energy to avoid confusion with the electric field E.

[xiv] For a series RLC circuit, $Q = (1/R)\sqrt{L/C}$

Great, but what is Q for our oscillating charge?

The average energy of a simple harmonic oscillator, you may recall, is always half kinetic and half potential, for a total of

$$\langle \mathcal{E} \rangle = \frac{1}{2} m \omega_o^2 x_o^2 \tag{1.45}$$

for an oscillator of mass m. If our oscillator is vibrating at its natural frequency ω_o, this gives us

$$\frac{1}{Q} = \frac{P}{\omega_o \mathcal{E}} = \frac{q^2 \omega_o^4 x_o^2}{12\pi\epsilon_o c^3} \left(\frac{1}{\frac{1}{2} m \omega_o^2 x_o^2} \right) \left(\frac{1}{\omega_o} \right) = \frac{q^2 \omega_o}{6\pi\epsilon_o c^3 m} \tag{1.46}$$

In terms of wavelength $\lambda_o = 2\pi c/\omega$,

$$\frac{1}{Q} = \frac{q^2}{3\epsilon_o m c^2 \lambda_o} = \left(\frac{q^2}{4\pi\epsilon_o m c^2} \right) \left(\frac{1}{\lambda_o} \right) \left(\frac{4\pi}{3} \right) = \frac{4\pi}{3} \frac{r_e}{\lambda_o} \tag{1.47}$$

The combination $r_e = q^2/4\pi\epsilon_o m c^2$ has units of length, and is known as the *classical electron radius* if the charges we are dealing with are individual electrons of charge $q = e$. The Q factor depends only on the ratio of the classical electron radius to the wavelength of radiation under consideration, which makes Q dimensionless overall as it must be. For $q = e$, the numeric value of r_e is

$$r_e = \frac{e^2}{4\pi\epsilon_o m c^2} \approx 2.8 \times 10^{-15}\,\mathrm{m} \tag{1.48}$$

The electron is, as far as we can tell, a point particle. The classical electron radius is based on an (incorrect) model of the electron, in which the electron is imagined as a uniform sphere of charge. In this model, r_e is roughly the size an electron would need to be for its rest energy to be completely due to electrostatic potential energy, ignoring quantum mechanics. We know now that the electron's rest energy is not electrostatic in nature, and quantum mechanics is required to understand the behavior of electrons on small distance scales. Still, r_e sets a semi-classical length scale for problems involving electrons, below which subtle quantum effects become extremely important.

Armed with this information, what is the Q value for a typical atom? For a sodium discharge lamp, the dominant emission is at a wavelength of about $\lambda = 600\,\mathrm{nm}$ (in the yellow region of the spectrum), so

$$Q = \frac{4\pi r_e}{\lambda_o} = \frac{3\epsilon_o m c^2 \lambda_o}{e^2} \sim 10^8 \tag{1.49}$$

The Q for a typical atom emitting visible light is $\sim 10^7 - 10^8$, meaning an atomic oscillator will oscillate for 10^7–10^8 radians or $\sim 10^7$ cycles before the energy is reduced by a factor $1/e \approx 1/2.718 \approx 0.37$.[xv] A wavelength of 600 nm implies a period of $\sim 10^{-15}$ s, so it takes about 10^{-8} s for the energy

[xv] Compare this to $Q = R\sqrt{C/L} \sim 10$–100 for typical circuit applications, possibly up to 10^6 for very precise circuits!

of a freely-oscillating atom in empty space to decay by a factor of $1/e$. It doesn't seem like much, but this is an eternity for an atom! Of course, for atoms in a solid or liquid, we have bonding and interactions between atoms to worry about, not to mention collisions, so there are additional sources of damping that decrease this time (and make it temperature-dependent).

Finally, we should note that the Q factor can be related to the *damping constant* γ of an oscillator, which is mathematically the coefficient of the 'viscous' force proportional to velocity:[xvi]

$$\frac{1}{Q} = 2\gamma \tag{1.50}$$

Evidently,

$$\gamma = \frac{q^2\omega_o}{12\pi\epsilon_o c^3 m} \tag{1.51}$$

In the following sections, we will derive the damping factor by considering the forces on an oscillating charge, but we will of course come to the same result.

Knowledge of the damping factor or Q factor also allow us to find the width of the resonance $\Delta\omega$, since $\Delta\omega = \omega_o/Q$. More useful is typically the linewidth $\Delta\lambda$ as a function of wavelength. Since $\lambda_o = 2\pi c/\omega_o$,[xvii] the variation in λ_o is

$$\Delta\lambda = \frac{2\pi c\Delta\omega}{\omega_o^2} = \frac{2\pi c}{Q\omega_o} = \frac{e^2}{3\epsilon_o mc^2} = \frac{e^2}{4\pi\epsilon_o mc^2}\frac{4\pi}{3} = \frac{4\pi r_e}{3} \tag{1.52}$$

For our sodium atom, this is amounts to $\Delta\lambda \sim 10^{-14}$ m. The relative linewidth (the "sharpness" of the line) is then

$$\frac{\Delta\lambda}{\lambda_o} = \frac{4\pi r_e}{3\lambda_o} \sim 10^{-8} \tag{1.53}$$

1.3.3 Charges in Circular Motion*

Another useful example of accelerated motion of charges is uniform circular motion.[xviii] If we put a charge q traveling at velocity v in a magnetic field B perpendicular to v, we know that the charge q will follow a circular path. We can find the radius of that path by noting that the magnetic force on the particle must provide the centripetal force to maintain the circular path:

$$qvB = \frac{mv^2}{r} \qquad \Longrightarrow \qquad r = \frac{mv}{qB} \tag{1.54}$$

The acceleration is just v^2/r, which is the magnetic force per unit mass:

For a laser cavity, one can achieve $Q \sim 10^{11}$.

[xvi] For a series RLC circuit, $\gamma = (R/2)\sqrt{C/L}$.

[xvii] Since $\Delta\lambda$ is an essentially an uncertainty, we must use propagation of uncertainty to find it. This is something you will encounter in your laboratory classes if you have not already; if it is unfamiliar, see, for example, `http://en.wikipedia.org/wiki/Propagation_of_uncertainty`. Basically, you differentiate both sides, giving $d\lambda = 2\pi c\, d\omega/\omega_o^2$, and presume small enough changes to turn the differentials into discrete changes.

[xviii] In this section we follow the treatment of Bekefi and Barrett[**?**] closely.

$$a = \frac{v^2}{r} = \frac{v^2 qB}{mv} = \frac{qvB}{m} \tag{1.55}$$

Of course, for circular motion we also know that the charge will repeat its motion with an angular frequency $\omega = v/r$:

$$\omega = \frac{v}{r} = \frac{a}{v} = \frac{qB}{m} \equiv \omega_c \tag{1.56}$$

This frequency ω_c is called the *cyclotron frequency*, and the radius of the path is called the *cyclotron radius*. With this in hand, we can use the Larmor formula to find the power radiated by the charge:

$$P = \frac{q^2 a^2}{6\pi\epsilon_o c^3} = \frac{q^2 \omega_c^2 v^2}{6\pi\epsilon_o c^3} \tag{1.57}$$

We can go even further than this, however: in uniform circular motion the potential and kinetic energies are equal. Since the kinetic energy is $K = \frac{1}{2}mv^2$, the total energy is $\mathcal{E} = mv^2$. We can then substitute $v^2 = \mathcal{E}/m$ into the Larmor equation above, and replace ω_c with qB/m:

$$P = \frac{q^2 \omega_c^2 \mathcal{E}}{6\pi\epsilon_o mc^3} = \frac{q^4 B^2 \mathcal{E}}{6\pi\epsilon_o m^3 c^3} \tag{1.58}$$

This is the radiated power, which is just the rate at which energy is lost: $P = -d\mathcal{E}/dt$, where the minus sign signifies that energy is being *lost* by the charge. Using this, we can find the energy of the charge as a function of time:

$$P = -\frac{d\mathcal{E}}{dt} = \frac{q^4 B^2}{6\pi\epsilon_o m^3 c^3} \tag{1.59}$$

$$\frac{d\mathcal{E}}{\mathcal{E}} = \frac{-q^4 B^2}{6\pi\epsilon_o m^3 c^3} \tag{1.60}$$

We can integrate both sides easily to solve for $\mathcal{E}$. For convenience, let $\tau \equiv 6\pi\epsilon_o m^3 c^3/q^4 B^2$, and let the charge's initial energy at $t = 0$ be $\mathcal{E}_o$. Then the energy as a function of time is:

$$\mathcal{E} = \mathcal{E}_o e^{-t/\tau} \tag{1.61}$$

The energy of the charge decays exponentially with time constant τ, which means the charge will *not* maintain circular motion, but will follow shrinking spiral path until it eventually stops. Incidentally, if we use the electric force of a nucleus in place of the magnetic force to keep an electron in circular motion, we have the classical planetary model of the atom ... which we can already see cannot possibly be stable. This problem is worked out in more detail in the following section.

1.3.4 Orbiting Charges: Classical Atoms*

In a hydrogen atom an electron of charge $-e$ orbits around a proton of charge $+e$. The electron must be constantly accelerating to stay in circular motion, which means it is radiating. This loss of energy implies a decaying orbit, which means after some time the electron will simply crash into the proton. An approach to finding out how long it will take might be as follows:

(a) Find the total energy E as a function of r, the distance between the electron and proton.
(b) Calculate the energy radiated per unit time as a function of r.
(c) Using $dr/dt = (dr/dE)(dE/dt)$, find the time it takes for a hydrogen atom to collapse from a radius of 10^{-9}m to a radius of 0.

The total energy is kinetic plus potential. The potential energy is that of two point charges e and $-e$ separated by a distance r. If we take the frame of reference that the (much heavier) proton is at rest, the kinetic energy is just that of the electron, to which we will assign mass m and velocity v:

$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_o r} \tag{1.62}$$

This equation has the electron velocity present, and we wish to find the energy as a function of radius only. We can eliminate the velocity by noting that the electric force between the proton and electron is constrained to equal the centripetal force required to maintain circular motion. That is,

$$\frac{-e^2}{4\pi\epsilon_o r^2} = -\frac{mv^2}{r} \qquad \Longrightarrow \qquad mv^2 = \frac{e^2}{4\pi\epsilon_o r} \tag{1.63}$$

Substituting into our first equation,

$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\epsilon_o r} = \frac{e^2}{8\pi\epsilon_o r} - \frac{e^2}{4\pi\epsilon_o r} = -\frac{e^2}{8\pi\epsilon_o r} \tag{1.64}$$

Just like gravitational orbits, the total energy is half of the potential energy. Given that the electron is in circular motion, it is accelerating, which means it must be radiating. The Larmor formula gives us the average radiated power, or energy per unit time:

$$\frac{dE}{dt} = -\frac{e^2 a^2}{6\pi\epsilon_o c^3} \tag{1.65}$$

Here we have inserted the minus sign because we know that the electron is losing energy by radiating. The acceleration a can be found from our force balance above, diving through by mass m:

$$a = -\frac{v^2}{r} = -\frac{e^2}{4\pi\epsilon_o m r^2} \tag{1.66}$$

Using the right-most form, we can find the power in terms of radius alone:

$$\frac{dE}{dt} = -\frac{e^2 a^2}{6\pi\epsilon_o c^3} = -\frac{e^2}{6\pi\epsilon_o c^3}\left(\frac{e^2}{4\pi\epsilon_o m r^2}\right)^2 = -\frac{e^6}{96\pi^3\epsilon_o^3 m^2 c^3 r^4} \tag{1.67}$$

If the electron is radiating, it is losing energy, which means its orbit must be decaying. With the power in hand, we can calculate the rate at which the radius of the electron's orbit decays and figure out how long such an atom would be stable. Using the chain rule

$$\frac{dr}{dt} = \frac{dr}{dE}\frac{dE}{dt} = \frac{dE}{dt}\Big/\frac{dE}{dr} \tag{1.68}$$

Since dE/dt is the power we just found, we need only dE/dr:

$$\frac{dE}{dr} = \frac{d}{dr}\left(-\frac{e^2}{8\pi\epsilon_o r}\right) = \frac{e^2}{8\pi\epsilon_o r^2} \tag{1.69}$$

Putting it together,

$$\frac{dr}{dt} = \frac{dE}{dt}\Big/\frac{dE}{dr} = -\frac{e^6}{96\pi^3\epsilon_o^3 m^2 c^3 r^4}\left(\frac{8\pi\epsilon_o r^2}{e^2}\right) = -\frac{e^4}{12\pi^2\epsilon_o^2 m^2 c^3 r^2} = -\left(\frac{e^4}{12\pi^2\epsilon_o^2 m^2 c^3}\right)\frac{1}{r^2} \tag{1.70}$$

For convenience, let $C = \frac{e^4}{12\pi^2\epsilon_o^2 m^2 c^3}$. This hideous combination is just a constant anyway, lumping it all together means we just have to keep track of one constant instead of 6. Our equation then reads

$$\frac{dr}{dt} = -\frac{C}{r^2} \tag{1.71}$$

This equation is separable[xix]:

$$r^2\, dr = -C\, dt \tag{1.72}$$

Integrating both sides, and noting that we start at time $t = 0$ at radius $r_i = 10^{-9}$ m and end at time t with radius zero,

$$\int_{r_i}^{0} r^2\, dr = -\frac{1}{3}r_i^3 = \int_0^t -C\, dt = -Ct \tag{1.73}$$

$$t = \frac{r_i^3}{3C} \tag{1.74}$$

Substituting our definition of C, the time for the electron to reach the proton is

$$t = \frac{4\pi^2\epsilon_o^2 m^2 c^3}{e^4} r_i^3 \tag{1.75}$$

[xix] If we close our eyes and manipulate the differentials like fractions, we would cross multiply to separate the equation.

With the given radius of $r_i = 10^{-9}\,\mathrm{m}$, $t \sim 10^{-7}\,\mathrm{s}$. Using a more realistic radius for the lowest energy state of a hydrogen atom, $r_i \approx 5\times10^{-11}\,\mathrm{m}$, one finds $t \sim 10^{-11}\,\mathrm{s}$. Moral of the story: classical atoms are not stable.

1.3.5 Radiation Reaction Force

From classical electrodynamics, we know that accelerating charges, such as oscillating charges, radiate electromagnetic waves, and therefore lose energy. If the oscillating charge is losing energy, it is also losing amplitude, and thus the radiation loss by the charge amounts to an effective damping force. In effect, the act of radiating acts as a recoil force on the accelerating charge, or a dissipation mechanism in some ways similar to viscous drag on a mechanical oscillator. The Larmor formula derived above relates the radiated power to the acceleration of the charge:

$$P = \frac{e^2 a^2}{6\pi\epsilon_o c^3} \tag{1.76}$$

The power of a mechanical system can be found in general from a knowledge of force and velocity:

$$P = \int \vec{F}\cdot\vec{v}\,dt \tag{1.77}$$

Let us consider the power emitted by our oscillator from time t_1 to time t_2, and let this time interval correspond to exactly one period of motion for our oscillator, i.e., $t_1 - t_2 = T = 1/f$. We will consider our oscillating charge to be a simple point charge of mass m and charge e with a natural resonance frequency of $\omega_o = 2\pi f_o$.[xx] Conservation of energy dictates that the power radiated away by the charge integrated over time must equal the mechanical power lost by the oscillator:

$$0 = \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt + \int_{t_1}^{t_2} P\,dt \qquad \text{or} \qquad \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = -\int_{t_1}^{t_2} P\,dt \tag{1.78}$$

Here we have restricted ourselves to non-relativistic velocities ($v \ll c$) since we used the classical form of momentum for mechanical power and force. Using Eq. 1.37, and noting $a = dv/dt$,

$$\int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = -\int_{t_1}^{t_2} P\,dt = -\int_{t_1}^{t_2} \frac{e^2a^2}{6\pi\epsilon_o c^3}\,dt = -\int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3}\frac{d\vec{v}}{dt}\cdot\frac{d\vec{v}}{dt}\,dt \tag{1.79}$$

We can integrate by parts, yielding[xxi]

[xx] This is equivalent to saying our mass m is connected to a spring of spring constant k, if you like.

[xxi] Recall $\int f\frac{dg}{dx}\,dx = fg - \int g\frac{df}{dx}\,dx$. Use $f = g = dv/dt$.

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d\vec{v}}{dt} \cdot \vec{v} \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v}\, dt \tag{1.80}$$

Since we are integrating over a full cycle of oscillation, the first term vanishes because $\frac{d\vec{v}}{dt} \cdot \vec{v}$ has the same value for equivalent points in the cycle of oscillation. Thus,

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v}\, dt \tag{1.81}$$

We can readily identify

$$\vec{F} = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^3\vec{x}}{dt^3} \tag{1.82}$$

This is the effective damping force acting the oscillating charge due to the fact that it is radiating. This "recoil" force is known as the *Abraham-Lorentz force.* Physically, the emitted radiation carries away momentum (since we know EM radiation carries momentum), and conservation of momentum dictates that the charge must be pushed in the direction opposite the direction of the emitted radiation. This is an unusual force, in that the charge is feeling a force in response to its own radiation! Essentially, we have just calculated one special case of the effect a charge has on itself - an odd problem to consider, in light of what we know of Newton's third law, but it is this problem which led to the development of quantum electrodynamics (QED), perhaps the most accurately-tested theory in all of physics.

1.3.6 Equation of motion for an oscillating charge

Our oscillating charge will experience a damping force due to the radiation it emits, and this damping force will act on the oscillatory motion in much the same way as a viscous fluid drag would on a mechanical oscillator. Not *exactly* the same, but within certain (reasonable) limits, we can reduce the problem of our oscillating charge to the familiar one of a damped harmonic oscillator.[xxii]

Without damping, the equation of motion for a simple harmonic oscillator of resonant frequency ω_o is[xxiii]

$$F = ma = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} = -kx = -m\omega_o^2 x \tag{1.83}$$

In the present situation, we must also include the radiation reaction force derived above, which acts as the same direction as the restoring force: $m\omega_o^2 x$:

[xxii] In this section we follow portions of the treatment by Feynman[?].

[xxiii] Again, for a mass-spring system, $\omega_o = \sqrt{k/m}$.

$$F = m\frac{d^2x}{dt^2} = -m\omega_o^2 x - \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3}$$
$$0 = m\frac{d^2x}{dt^2} + m\omega_o^2 x + \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3} \tag{1.84}$$

With the radiation reaction force present, the amplitude of oscillation will decay with time, as a would be the case for a mechanical oscillator (though in a somewhat more complicated way, given that the form of the damping force is different). We are not interested in the isolated case of a single oscillator, however, but rather the case where the oscillator is interacting with an electric field, particularly that due to thermal radiation in subsequent sections. That is, we wish to consider a *driven* oscillator.

The simplest possible case would be to consider what happens when our oscillating charge is exposed to a monochromatic electric field, i.e., an electric field which varies sinusoidally with time with a single frequency $\omega = 2\pi f$:

$$|\vec{E}| = E_o \cos\omega t \tag{1.85}$$

where in general $\omega \neq \omega_o$, i.e., the frequency of the driving electric field is not necessarily identical to the resonance frequency of the oscillating charge. This time-varying electric field, the electric portion of an EM wave, will produce a time-varying force $e|\vec{E}|$ on our charge, which is the driving force for our oscillator. Adding this driving force to our already-damped oscillator (Eq. 1.84):

$$m\frac{d^2x}{dt^2} + \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3} + m\omega_o^2 x = eE_o \cos\omega t \tag{1.86}$$

This is a tough equation, more than we wish to handle. What we would really like is to somehow make this equation look like the driven harmonic oscillator we already know and love.[xxiv] But what to do with that ugly third derivative?

The situation is not so bad as it seems. In most cases of interest, the radiation resistance force is small compared to the restoring force giving rise to the oscillation (the atomic bonds).[xxv] In this case of small damping, the acceleration is *approximately* the same as it is without damping, or $a \sim \omega_o^2 x$. If this is the case,

$$\frac{d^2x}{dt^2} \sim \omega_o^2 x \qquad \text{or} \qquad \frac{d^3x}{dt^3} = \frac{da}{dt} \sim \omega_o^2 \frac{dx}{dt} \tag{1.87}$$

The basic idea is this: the damping term with the third derivative is small in Eq. 1.86, so for

[xxiv] See `http://en.wikipedia.org/wiki/Harmonic_oscillator` for a quick review.

[xxv] We can make an order-of-magnitude estimate from Eq. 1.82: presuming an amplitude of vibration of 0.1 nm (very large for an atom!), incident red light ($\omega_o/2\pi = f_o \sim 5 \times 10^{14}$ Hz), and a maximum acceleration of $\omega_o^2 A$ over a time of $1/f_o \approx 10^{-15}$ s, we find a force in the 10^{-18} N range. Using as an example the force constant for an HCl molecule, $k \sim 500$ N/m, and a displacement of 0.1 nm from equilibrium we find a restoring force of order 10^{-8} N, a comfortable ten orders of magnitude larger than the radiation resistance. This is consistent with our estimate of $Q \sim 10^8$ for an oscillating atom in empty space, another way of saying the dissipation is small.

that term we will use the substitution above as a good approximation. The other terms we will leave alone, since we have no reason to presume they are small, and we know how to deal with them anyway. This gives us:

$$m\frac{d^2x}{dt^2} + \frac{e^2\omega_o^2}{6\pi\epsilon_o c^3}\frac{dx}{dt} + m\omega_o^2 x = eE_o\cos\omega t \tag{1.88}$$

$$\frac{d^2x}{dt^2} + \frac{e^2\omega_o^2}{6\pi\epsilon_o mc^3}\frac{dx}{dt} + \omega_o^2 x = \left(\frac{eE_o}{m}\right)\cos\omega t \tag{1.89}$$

If we define a "damping constant" γ

$$\gamma = \frac{e^2\omega_o}{12\pi\epsilon_o mc^3} \tag{1.90}$$

we can make our equation of motion just like that of a driven harmonic oscillator with a viscous damping proportional to velocity, or an LC resonant circuit with resistance included. This equation we know the solution to already.[xxvi]

$$\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{eE_o}{m}\cos\omega t \tag{1.91}$$

$$\Longrightarrow \qquad x(t) = A\cos(\omega t + \varphi) \tag{1.92}$$

The table below shows the analogous quantities for series and parallel RLC circuits and a mechanical oscillator

	Series RLC	**Parallel RLC**	**Mechanical**
restoring	inverse capacitance $1/C$	inverse inductance $1/L$	spring constant k
"mass"	inductance L	capacitance C	mass m
friction	R	$1/R$	damping coefficient c
damping γ	$\frac{1}{2}R\sqrt{C/L} = \frac{1}{2}RC\omega_o = R/2L\omega_0$	$\frac{1}{2R}\sqrt{L/C} = \frac{1}{2R}L\omega_o = 1/2RC\omega_o$	c/m
ω_o	$\sqrt{1/LC}$	$\sqrt{1/LC}$	$\sqrt{k/m}$
$Q = 1/2\gamma$	$\frac{1}{R}\sqrt{L/C}$	$R\sqrt{C/L} = RC\omega_o = R/L\omega_o$	$m/2c$

The steady-state solution to this equation given above can be found readily with complex exponentials; we will presume you have done this sort of thing before. If not ... you will. Many, many times. The solution gives us the amplitude A and phase φ of vibration of the oscillator as a function of the driving frequency ω and the damping constant γ:

[xxvi] Note that this is the same damping constant we found in Eq. 1.51!

$$A(\omega) = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \tag{1.93}$$

$$\varphi = \tan^{-1}\left(\frac{2\omega\omega_o\gamma}{\omega^2 - \omega_o^2}\right) \tag{1.94}$$

From the amplitude, one can also find the resonance frequency (where $A(\omega)$ is maximal) as $\omega_r = \omega_o\sqrt{1-2\gamma^2}$, which for small damping reduces[xxvii] to $\omega_r \approx \omega_o(1-\gamma^2) \approx \omega_o$. From the phase equation, we can see that for low driving frequencies, $\omega < \omega_o$, the phase angle is small and the charge will oscillate in sync with the driving field. However, when $\omega > \omega_o$, the displacement is in the opposite direction from the driving force, 180° degrees out of phase with the field. Consequently, the amplitude strongly decreases above ω_o, and more gradually below ω_o. The amplitude displays a sharp peak in the region where the driving frequency matches the oscillator's resonance frequency, $\omega = \omega_r$, as shown in Fig. 1.9.

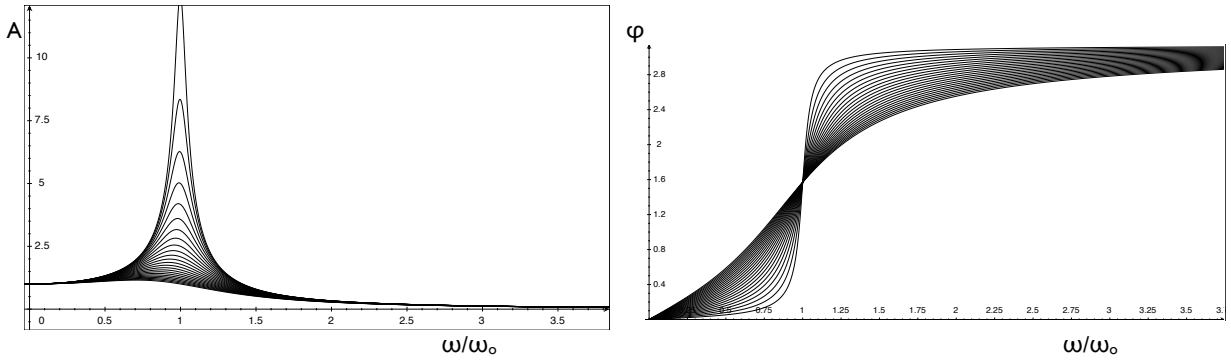

Figure 1.9: (upper) *Relative amplitude of oscillation versus driving frequency with* γ *ranging from* 0.04 *(top curve) to* 0.5 *(bottom curve) in steps of* 0.02. *The linewidth of the resonance curve is* $\omega_o/2Q$. **(lower)** *Phase in radians versus driving frequency with* γ *ranging from* 0.04 *(sharpest curve) to* 0.5 *(smoothest curve) in steps of* 0.02.

Given the amplitude, we can also find the potential energy of the oscillator, $U = \frac{1}{2}m\omega_o^2 A^2$. Averaged over a whole cycle, the kinetic and potential energies of the oscillator are the same, so the total average energy is just $m\omega_o^2 A^2$. We will make use of this in the following sections.

As a quick sanity check on our answer, we can check that our result makes sense reducing our result to the case of no damping, $\gamma \to 0$, which gives

$$A(\omega) = \frac{eE_o}{m(\omega_o^2 - \omega^2)} \qquad (\gamma \to 0) \tag{1.95}$$

This is just what we expect for a driven oscillator without damping. If we remove the periodicity of the driving force ($\omega \to 0$) just have a free oscillator in a static electric field:

$$A = \frac{eE_o}{m\omega_o^2} \qquad (\omega \to 0) \tag{1.96}$$

[xxvii] Using $(1+x)^n \approx 1 + nx$.

Note that this is the same result one gets from a force balance, $m\omega_o^2 A = kA = eE$.

What have we learned over all? Our charged oscillator is driven by a periodic electric field, and this field 'feeds' energy into the oscillator, which is in turn drained away by radiation damping.[xxviii] That is, the charge absorbs energy from the electric field, and reemits it as radiation at the same frequency. This leads to a steady-state equilibrium, in which the energy gained from the field balances the energy lost by radiation.

More importantly, we are slowly building up a model of the interaction of radiation and matter. We can imagine that our oscillating charges are not bare electrons, but perhaps the most weakly-bound electrons in the atoms of a gas. What we are really doing is trying to figure out how radiation – light – is emitted and absorbed by matter.

1.3.7 Scattering of Light*

What if instead of a single oscillating charge in a single atom, we have many? In a nice crystal, we would expect that we have constructive and destructive interference of emitted radiation due to the regular, periodic arrangement of atoms. If we consider a *random* collection of atoms with oscillating charges, however, overall there is no net constructive or destructive interference, and the total intensity is just the sum of the intensities of all the individual atoms. Even in a regular crystal, random thermal motion of the atoms means that at any given moment strict periodicity is broken, and so the strict condition for interference is also broken. Essentially, we assume that all the atoms incoherently emit radiation, and so we can just figure out the radiative properties of a single atom and multiply by the number of atoms. Physically, what we have is incident light in a single direction falling on an atom, and being reemitted over a range of angles, or what we usually call *scattering*.[xxix]

What we wish to figure out now is what happens when an incident beam of light (an EM wave) strikes at atom. We know the incident light beam has an electric field component like $\vec{E} = \vec{E}_o e^{i\omega t}$, and when it strikes an atom an electron in the atom will feel a periodic force $q\vec{E}$ and begin to vibrate up and down. Thus, the charge accelerates, and re-radiates some of the energy it received from the incident electric field. This is *scattering* of light, and more importantly, it is again our driven harmonic oscillator. We know the amplitude of vibration will be given by Eq. 1.93, so the position as a function of time is:

$$x(t) = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \cos(\omega t + \varphi) \tag{1.97}$$

Since we know that we have a large Q factor for an isolated atom, we will for the moment neglect damping ($\gamma \to 0$) to simplify matters. We could also try to take into account that the electron might

[xxviii] Sort of in the same way that in pushing a person on a swing you are the driving force, feeding periodic energy to maintain the oscillations. The resultant amplitude is largest when your driving frequency matches the natural frequency of the person and the swing, and if your pushes are out of phase with the swing, the amplitude of oscillation is dramatically reduced.

[xxix] In this section we follow the treatment of Feynman[**?**] closely.

act as an oscillator with several different frequencies, but we will also neglect this complication. Without damping, we have an amplitude

$$x(t) = \frac{eE_o \cos \omega t}{m\left(\omega_o^2 - \omega^2\right)} \tag{1.98}$$

From this, we can find the acceleration and calculate the power re-radiated by the charge in any given direction using Eq. 1.39. A somewhat simpler task is to just find the total emitted power. We can use Eq. 1.37 and the acceleration determined from $x(t)$ above, which is in fact what we already did in deriving Eq. 1.41. All we need to do is replace the the amplitude for a free harmonic oscillator x_o with the amplitude of our driven harmonic oscillator given by Eq. 1.93 with the damping γ set to zero:

$$P = \frac{e^2\omega^4 A^2}{12\pi\epsilon_o c^3} = \frac{e^2\omega^4}{12\pi\epsilon_o c^3}\frac{e^2E_o^2}{m^2\left(\omega_o^2 - \omega^2\right)^2} = \left(\frac{1}{2}\epsilon_o E_o^2\right)\left(\frac{e^4}{6\pi\epsilon_o^2 c^3 m^2}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} \tag{1.99}$$

If we substitute for the classical electron radius (Eq. 1.48), we find

$$P = \left(\frac{1}{2}\epsilon_o E_o^2\right)\frac{e^4}{16\pi^2\epsilon_o^2 m^2 c^4}\left(\frac{8\pi c}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} = \left(\frac{1}{2}\epsilon_o E_o^2\right)\left(\frac{8\pi r_e^2 c}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} \tag{1.100}$$

The most important result thus far is that the scattered energy goes as the *square* of the field, or as the (time-averaged) energy density of the incident field which is $\frac{1}{2}\epsilon_o E_o^2$. Since the *intensity* of electromagnetic radiation goes as E^2, the scattered radiation intensity is proportional to the incident radiation intensity. Basically: the brighter the source, the brighter the scattered light!

We can look at this in another way, however. Say we have light going through a surface of area σ. How much radiant energy passes through that surface in a given time t? It would be the energy density of the field, multiplied by the area σ, multiplied by the distance that light can travel during time t, or ct. The rate at which energy passes through the surface, the power transmission, is then just that energy divided by t, or $P = \frac{1}{2}\epsilon_o c E_o^2 \sigma$. Comparing that to what we have in Eq. 1.100 already, we notice for the scattered light

$$P = \frac{1}{2}\epsilon_o c E_o^2 \sigma = \left(\frac{1}{2}\epsilon_o E_o^2\right)\left(\frac{8\pi r_e^2 c}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} \tag{1.101}$$

$$\Longrightarrow \qquad \sigma = \left(\frac{8\pi r_e^2}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} \tag{1.102}$$

Indeed, the right-hand side does have units of area! What is the meaning of this area? An atom scatters a certain total amount of radiation, which would then end up falling on a certain area, and it is this area σ that we just found. Our identification of σ above amounted to taking the ratio of the total energy scattered per second to the incident energy per square meter:

$$\sigma = \frac{P}{\frac{1}{2}\epsilon_o c E_o^2} = \frac{\text{total scattered energy per second}}{\text{incident energy per square meter per second}} \tag{1.103}$$

The area σ is usually called a *scattering cross section*, and it is a concept that is used frequently in physics. The idea is that the energy intercepted by the area σ is the same as that scattered by the atom. In other words, it is a measure of how much of the beam we would need to block to scatter away as much of the incident light as the atom does. In that way it is a sort of characteristic 'size' associated with scattering, and we could compare the these sizes for different scattering mechanisms to gauge their relative strengths[xxx] There isn't any real physical area to speak of – just oscillating point charges – but the *effect* is the same as if we made a tiny beam block of area σ to scatter away some of the incident light. Based on the definition above, and the fact that $\frac{1}{2}\epsilon_o E_o^2$ is just the average energy per unit volume of the incident electric field, the scattered power must be

$$P_{\text{scattered}} = \sigma c \langle u_E \rangle = \sigma I_{\text{incident}} \tag{1.104}$$

where I_{incident} is the *irradiance*, a common measure of radiation intensity. Irradiance is the energy flux per unit area, averaged over one period of oscillation, and it can be found from $I = c\langle u_{\text{field}} \rangle$. This is a sensible result: the scattered intensity is proportional to the incident intensity, so again the brighter the source, the brighter the scattered light!

Incidentally, the cross section we've found does not include radiation damping. If we repeat our derivation above without neglecting damping,[xxxi] things are only slightly more complex, and it is clear that non-zero damping *reduces* the cross section, i.e., the atoms are less effective scatterers:

$$\sigma = \left(\frac{8\pi r_e^2}{3}\right) \frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2 + \left(2\gamma\omega\omega_o\right)^2} \tag{1.105}$$

What conclusions can we draw? One, the scattering depends strongly on ω. Since we have $\omega_o^2 - \omega^2$ in the denominator, the scattering cross-section becomes very large at the resonance frequency of an electron in an atom. This makes sense: the incident radiation can most efficiently transfer its energy to an electron when its frequency matches the resonance frequency, and at resonance the electron will most efficiently re-radiate. Two, the numerator of the cross-section grows as ω^4, meaning that it is much larger above resonance than below. Three, the energy dependence of light scattering explains why the sky is blue! The constituents of the atmosphere have their relevant resonance frequencies well in the ultraviolet. Visible light is at much lower frequencies, so we are looking at the cross section at frequencies below the resonant peak. In this regime, higher frequency blue light is scattered more than lower frequency red light owing to the larger cross section. What you're seeing when you look away from the sun is the light which is scattered more by the atmosphere, which is more blue than red light. This also means that ultraviolet light is absorbed even more

[xxx] The typical area unit used for scattering is the *barn*. It is commonly used in all fields of high energy physics to express the cross sections of scattering processes. A barn is $10^{-28}\,\text{m}^2$ ($100\,\text{fm}^2$), approximately the cross sectional area of a uranium nucleus. The term originated with American physicists during wartime research on the atomic bomb, scattering neutrons off of uranium nuclei. They described the uranium nucleus as "big as a barn".

[xxxi] Which really only amounts to replacing $\left(\omega_o^2 - \omega^2\right)^2$ with $\left(\omega_o^2 - \omega^2\right)^2 + \left(2\gamma\omega\omega_o\right)^2$

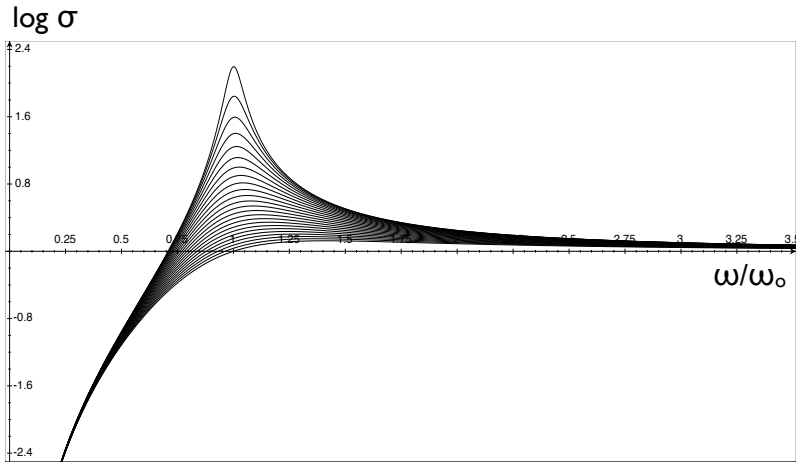

Figure 1.10: *Logarithm of the scattering cross section versus driving frequency with* γ *ranging from* 0.04 *(top curve) to* 0.5 *(bottom curve) in steps of* 0.02*. The width of the resonance curve at half maximum is* ω_o/Q*.*

strongly, which is a good thing.[xxxii] Mathematically, if $\omega \ll \omega_o$ and damping is negligible, then Eq. 1.105 reduces to

$$\sigma \approx \left(\frac{8\pi r_e^2}{3}\right)\frac{\omega^4}{\omega_o^4} \tag{1.106}$$

The cross section grows as ω^4 (or decreases as λ^{-4} if you like), so higher frequency (smaller wavelength) radiation is scattered much more effectively. This is known as Rayleigh scattering,[xxxiii] though our analysis has left out some details, such as the angular distribution of the radiation (which we could recover easily enough from our derivation of the Larmor formula) and polarizability of the scattering medium (which is just an overall multiplying factor).

[xxxii] Ozone is particularly good at absorbing ultraviolet light, hence the importance of the ozone layer in the atmosphere.

[xxxiii] See `http://en.wikipedia.org/wiki/Rayleigh_scattering` for more details on scattering and why the sky is blue.

2 Thermal Radiation

Finally, we are ready to address the subject of thermal radiation.[i] Our idea is the following: we know how to calculate the emission of radiation from oscillating charges, and how they scatter incident radiation generated by other charges. We will imagine that we have a hot object (say, a gas in a perfectly black box) made up of many identical atoms, each of which has electrons that can be induced to oscillate and radiate. Our hot atoms inside the box will acquire thermal energy, and random motion will be induced. This random motion will result in the atoms having many different frequencies of oscillation, which means that any given atom is being exposed to radiation with a wide range of frequencies added (incoherently) together. What we would like to do is figure out the energy emitted by a single atom in the box exposed to the radiation from all others over a spread range of frequencies. If we can figure out the energy re-emitted by a single atomic oscillator driven by thermally-induced radiation, we should be able to determine the spectrum of thermally-induced radiation since in principle we already know the amount of thermal energy present.

This might seem intractable at first, but we've already figured out the problem for a single incident frequency of light impinging on an oscillator in the previous section. In the case of non-zero but small damping, we can see from Fig. 1.9 that the only driving frequencies that really matter are those close to the resonance frequency of the oscillator $\omega \approx \omega_r \approx \omega_o$, only those frequencies give rise to a large amplitude of oscillation.

Using our previous results, for a given mode of oscillation at resonance frequency ω_o driven by radiation at frequency ω, the *total* energy of the oscillator is

$$U_{osc} = m\omega_o^2 A^2 = m\omega_o^2 \frac{e^2 E^2/m^2}{\left(\omega_o^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\omega_o^2} \tag{2.1}$$

If only frequencies near resonance, $\omega \approx \omega_o$ will lead to large amplitudes (and therefore significant radiated power), we can approximate the first part of the denominator in the equation above. First, a bit of factoring:

$$\left(\omega_o^2 - \omega^2\right)^2 = \left(\omega_o^2 - \omega^2\right)\left(\omega_o^2 - \omega^2\right) = \left(\omega_o - \omega\right)^2 \left(\omega_o + \omega\right)^2 \tag{2.2}$$

If $\omega \approx \omega_o$, then $\omega_o + \omega \approx 2\omega_o$, and

$$\left(\omega_o^2 - \omega^2\right)^2 \approx 4\omega_o^2 \left(\omega_o - \omega\right)^2 \tag{2.3}$$

This leads us to an expression for the oscillator energy as a function of the driving frequency of the incident radiation ω:

[i] In this section we follow the treatment by Fowler[**?**].

$$U_{osc} \approx \left(\frac{\omega_o^2}{m}\right) \frac{e^2 E^2}{4\omega_o^2 (\omega_o - \omega)^2 + 4\gamma^2 \omega_o^4} = \left(\frac{e^2 E^2}{4m}\right) \frac{1}{(\omega - \omega_o)^2 + \gamma^2 \omega_o^2} \tag{2.4}$$

where we have also used $\omega \approx \omega_o$ for the damping term in the denominator.

This is still for a single precise frequency of incident radiation ω, but we wish to sum over all incident frequencies to find the total energy of the oscillator. If U_{osc} is the energy of the oscillator at frequency ω, then $U_{osc}\, d\omega$ is the energy contained in the narrow frequency range $\omega \in [\omega, \omega + d\omega]$.[ii] Summing over all such frequency ranges $d\omega$ amounts to integrating $U(\omega)\, d\omega$ over that same interval. In the case of small damping, it really won't matter much if we integrate $U(\omega)$ only around the peak at ω_o or over all frequencies from 0 to ∞ since $U(\omega)$ only has appreciable weight in a narrow region around ω_o. Thus, for an oscillator driven by a wide range of possible frequencies of incident radiation, the total energy is:

$$U_{osc,tot} \approx \int_0^\infty \left(\frac{e^2 E^2}{4m}\right) \frac{1}{(\omega - \omega_o)^2 + (\gamma\omega_o)^2}\, d\omega = -\frac{e^2 E^2}{4m\gamma\omega_o} \tan^{-1}\left(\frac{\omega - \omega_o}{\gamma}\right)\Bigg|_0^\infty = \frac{\pi e^2 E^2}{8m\gamma\omega_o} \tag{2.5}$$

Actually, we have missed one important detail: in considering the possible frequencies of driving radiation along a given axis, we have two possible polarizations of radiation to consider (i.e., oscillations along two possible directions perpendicular to the incident light propagation), so we must multiply by two. Doing that and using our definition of γ from Eq. 1.90:

$$U_{osc,tot} = \frac{\pi e^2 E^2}{4m\gamma\omega_o} = \left(\frac{1}{2}\epsilon_o E^2\right) \frac{6\pi^2 c^3}{\omega_o^2} \tag{2.6}$$

The term in brackets on the right is once again the total energy per unit volume contained in the electric field (a.k.a., the energy density):

$$u_{field} = \frac{1}{2}\epsilon_o E^2 = U_{osc,tot} \frac{\omega_o^2}{6\pi^2 c^3} \tag{2.7}$$

What we have now is a relationship between the total energy of a single oscillating charge and the energy contained in the electric field it is immersed in. This is of course only for a single component of the field, since we have thus far considered oscillations only in a single plane resulting from radiation incident from a single direction. The other two directions of the field and planes of oscillation will give the same result if the system is homogeneous and isotropic, so to account for the other directions we simply multiply by three:

$$u_{field} = \frac{\omega_o^2}{2\pi^2 c^3} U_{osc,tot} = \frac{2f^2}{c^3} U_{osc,tot} \tag{2.8}$$

You can think of this result in a slightly different way: the quantity $u_{field}(\omega)\, d\omega$ gives the

[ii] Think about slicing the area under the $U(\omega)$ curve into tiny rectangles of width $d\omega$.

energy per unit volume for radiation with angular frequency ω in the frequency range $[\omega, \omega + d\omega]$. This is a crucial result: what is says is that if we can find the total energy of a given oscillator by other means, we automatically know the energy contained in the radiation field at a given frequency. Clearly, the idea is that we should use thermodynamics to find the energy of an oscillator at temperature T and use it to find the radiation energy density and spectrum.

Here is where trouble starts!

2.1 Rayleigh-Jeans Law

From classical thermodynamics, we know that each oscillator has an average energy $\langle U_{osc,tot}\rangle = k_B T$ at a temperature T independent of the oscillator's frequency. Thus, for an oscillator at a given frequency, we would expect the energy density of the electric field to be

$$\langle u_{field}\rangle = \frac{2f^2}{c^3}\langle U_{osc,tot}\rangle = \frac{2k_B T f^2}{c^3} \tag{2.9}$$

A common measure of radiation intensity is the *irradiance* (often called simply "intensity"), the energy flux per unit area, averaged over one period of oscillation, and it can be found from $I = c\langle u_{field}\rangle$,[iii] giving

$$I = \frac{2k_B T f^2}{c^2} \tag{2.10}$$

This is the famous Rayleigh-Jeans law, which says that the energy per unit volume of thermally emitted radiation should scale as T and f^2. It agrees reasonably well with experimental results at low frequencies (long wavelengths), but strongly disagrees at high frequencies. In fact, it predicts that the energy density should be arbitrarily large as frequency increases! Worse, since the energy is only an average it implies that *any* source of thermal energy should contain at least some high-frequency radiation. Since we know everyday hot objects don't emit X-rays, this is a problem, often called the "ultraviolet catastrophe" among those prone to hyperbole. It is no catastrophe in the grand scheme of things, it just means that our model has gone horribly wrong somewhere. In particular, it has gone wrong by assuming that oscillators of any frequency receive the same amount of energy.

2.2 Planck's Hypothesis

Where did we go wrong? We assumed that all oscillators receive the same $k_B T$ worth of thermal energy, no matter what their frequency of oscillation. This seems odd! Planck's ad hoc resolution to the problem was to assume that perhaps the oscillators cannot emit arbitrary amounts of energy, but only multiples of a smallest indivisible unit of energy. That is, we assume energy only comes

[iii] Intensity is power per unit area going through a patch of surface, which is just the energy density multiplied by the velocity at which the energy is moving through a given area, c. See `http://en.wikipedia.org/wiki/Intensity_%28physics%29`

in discrete bundles, rather than arbitrary amounts. This isn't totally crazy – the resonant standing modes of a vibrating string only have certain allowed energies, after all, owing to the geometric boundary conditions imposed. Perhaps energy is similarly discrete, owing to some yet-unforeseen boundary conditions on the smallest scales?[iv]

Specifically, let us imagine that oscillators only emit energy in small bundles proportional to their frequency. After all, it makes some sense that the faster the oscillation, the more energy emitted by the oscillator. Planck proposed that energy is *quantized* and only comes in units of $\Delta E = hf$, where h is now known as *Planck's constant*. We now know that

$$h \approx 6.626 \times 10^{-34}\,\mathrm{J}\cdot\mathrm{s} = 4.135 \times 10^{-15}\,\mathrm{eV}\cdot\mathrm{s} \tag{2.11}$$

Planck's constant is *tiny*, which explains why we didn't notice the discretization of energy sooner – the "graininess" of energy is far too small to be noticed on the scale of everyday energies ... but it is kind of a big deal for tiny things like atoms!

What this implies is that the allowed energies of our oscillators can only take on discrete integer multiples of hf. Thus, an oscillator can have energies of $E = \{0, hf, 2hf, 3hf, \ldots\}$ but not $E = 1.5hf$. The particular energy of an oscillator at any given moment can then simply be indexed by an integer n telling us how many units of energy it has: $E = nhf$, $n = \{0, 1, 2, 3, \ldots\}$. More formally, we could state the hypotheses of Planck as:[v]

1. Each oscillator *absorbs* energy from the radiation field in a continuous fashion, following classical electrodynamics.
2. An oscillator can *radiate* energy only in *exact integral multiples* of energy proportional to its frequency. When an oscillator does radiate, it radiates *all* of its energy.
3. The radiation or non-radiation of an oscillator when it possesses an integral number of energy units is entirely governed by statistical chance. The ratio of the probability of nonemission to the probability of emission is proportional to the intensity of the radiation exciting the oscillator.

The first part we have already dealt with, the excitation of the oscillators by the radiation bath they are immersed in. The second is Planck's discretization hypothesis sketched above. The third point we may derive from classical thermal physics along with the discretization hypothesis, and taken together, we will be able to rescue our model of radiation.

What we must now ask ourselves is for a given temperature T, what is the average energy of our oscillators given these new constraints? At $T = 0$, absolute zero, there would be no thermal energy, so all oscillators would have an energy of 0 and occupy the state $n = 0$. At any nonzero temperature, our oscillators will be distributed over levels of various n, our task is to figure out how they are distributed. Since thermal energy is random, but with a well-defined mean, some oscillators will

[iv] In this section we follow portions of the treatment by Feynman[**?**].

[v] Following Leighton[**?**] Ch. 2.

have the lowest possible energy, most in between, and a few will have relatively high energies. At low temperatures, when the thermal energy is small, most oscillators will have zero energy, and as temperature increases, more and more oscillators will be able to gain the right amount of thermal energy to have energies of hf, $2hf$, $3hf$, etc.

Given these equally-spaced energy levels, we can use the Boltzmann factor to find the probability that a given oscillator has a particular energy. Recall that the Boltzmann factor tells us the probability $P(E)$ of a given particle having an energy E at a temperature T:

$$P(E) = e^{-E/k_B T} \tag{2.12}$$

where $k_B = 1.38 \times 10^{-23}\,\text{J/K} = 8.617 \times 10^{-5}\,\text{eV/K}$ is Boltzmann's constant. In the present case, the lowest possible energy is for $n = 0$, corresponding to $E = 0$. Let's say we have many oscillators, N_{tot} in total. We have an infinity of possible energy levels, $n = \{0, 1, 2, \ldots\}$ corresponding to $E_n = \{0, hf, 2hf, \ldots\}$. Let the number of oscillators in each of these levels be $N_n = \{N_0, N_1, N_2, \ldots\}$. What is the average energy of all the oscillators? It must still be the total energy of all oscillators divided by the number of oscillators. The total energy is just the sum over all levels of the number of oscillators in each level times the energy of that level.

$$\langle E \rangle = \frac{\text{total } E}{\text{number of oscillators}} = \frac{\sum (\text{number per level})\,(\text{energy of level})}{\text{number of oscillators}} \tag{2.13}$$

The number of atoms in any given level is just the number of oscillators in total times the probability that a given level is occupied. The Boltzmann factor gives us the latter quantity. If there are N_0 atoms in the lowest energy level $n = 0$, for a level n the number of oscillators with that energy is

$$N_n = N_0 e^{-E_n/k_B T} \tag{2.14}$$

The total energy of all oscillators together is just summing up the number in each level times the energy of that level:

$$E_{tot} = \sum_{n=0}^{\infty} N_n E_n = \sum_{n=0}^{\infty} N_0 e^{-nhf/k_B T} nhf \tag{2.15}$$

This is not so bad a sum as it looks; define a new variable $x = e^{-hf/k_B T}$, then the sum becomes a simple geometric series:

$$E_{tot} = \sum_{n=0}^{\infty} N_0 x^n nhf = N_0 hf \sum_{n=0}^{\infty} n x^n = N_o hf \frac{x}{(1-x)^2} \tag{2.16}$$

The number of oscillators in total is found by summing the number in each level, which results in another well-known series:

$$N_{tot} = \sum_{n=0}^{\infty} N_n = \sum_{n=0}^{\infty} N_0 e^{-E_n/k_B T} = N_o \sum_{n=0}^{\infty} x^n = N_o \frac{1}{1-x} \tag{2.17}$$

Thus, the average energy is

$$\langle E \rangle = \frac{E_{tot}}{N_{tot}} = hf \frac{x}{(1-x)^2}(1-x) = hf \frac{x}{1-x} \tag{2.18}$$

Recalling our definition of x,

$$\langle E \rangle = hf \frac{e^{-hf/k_B T}}{1 - e^{-hf/k_B T}} = \frac{hf}{e^{hf/k_B T} - 1} \tag{2.19}$$

This is the famous Planck formula for the average energy of the oscillators. It does not suffer from the divergences at high frequencies, and it is very different from the classical result $\langle E \rangle = k_B T$. In Fig. 2.1 below we have plotted $\langle E \rangle$ versus f for various values of $h/k_B T$, just to give you an idea of how $\langle E \rangle$ behaves compared to the classical prediction of $\langle E \rangle = k_B T$, independent of frequency.

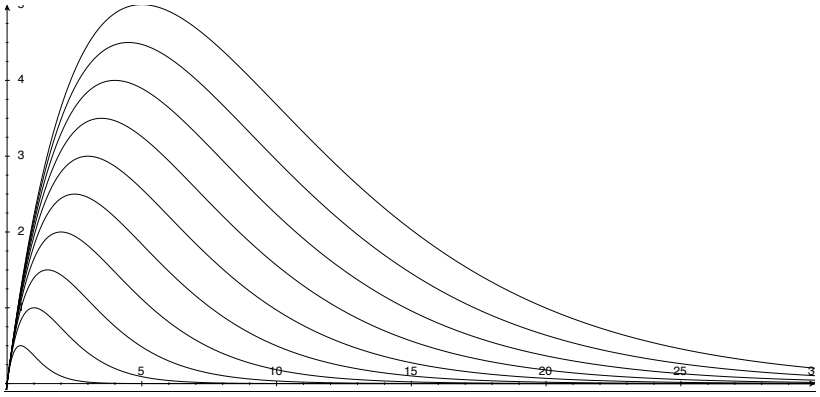

Figure 2.1: $\langle E \rangle$ *versus* f *for various values of* $h/k_B T$.

Qualitatively, we see that at high temperatures, the oscillators are spread out over a wide range of energy levels, while at low temperatures they tend to occupy only the lowest levels. When $k_B T \gg hf$, i.e., when the thermal energy is much larger than the discrete spacing between energy levels, the discreteness of energy becomes unimportant, and our prior result holds at low frequencies or high temperatures. (You can show that this result is recovered in the case that $f \to 0$ or $T \to \infty$.) This is again a reason why the discreteness of energy took so long to notice: unless the system of interest is at very low temperatures, or very high frequency radiation is involved, the spacing of allowed energies is too small compared to the random thermal energy to be noticed.

2.3 The Radiation Spectrum

Armed with a new expression for the average energy of the oscillators, we can immediately apply our previous result:

$$\langle u_{\text{field}} \rangle = \frac{2f^2}{c^3} \langle U_{\text{osc,tot}} \rangle = \frac{2f^2}{c^3} \frac{hf}{e^{hf/k_B T} - 1} = \frac{2hf^3}{c^3} \frac{1}{e^{hf/k_B T} - 1} \tag{2.20}$$

The intensity (average power per unit area) is thus

$$I = c\langle u_{\text{field}} \rangle = \frac{2hf^3}{c^2} \frac{1}{e^{hf/k_B T} - 1} \tag{2.21}$$

This reproduces, with amazing accuracy, the observed emitted radiation energy per unit volume versus frequency. To find the intensity as a function of wavelength, the change of variable requires evaluating

$$I'(\lambda, T) = I(f, T) \left| \frac{df}{d\lambda} \right| = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda k_B T} - 1} \tag{2.22}$$

At this point, your textbook takes over fairly well in discussing the main features of thermal ("blackbody") radiation. In the problems below we derive Wein's displacement law relating the wavelength of peak radiation emission to temperature, show that the *total* emitted power over all wavelengths scales as T^4 (related to the Stefan-Boltzmann law), and consider some everyday near-blackbody sources (the sun, incandescent light bulbs).

3 Electromagnetic waves in solids

Now that we have a reasonable handle on radiation and scattering of electromagnetic waves in free space, it is useful to consider how electromagnetic waves behave in matter. This will lead us to descriptions of reflection and refraction, as well as the frequency-dependent behavior of dielectric and conducting materials. As a starting point, we will quickly review the response of dielectrics to an electric field.

We must make a few restrictions for these problems to be tractable. First, we will consider only *isotropic* materials, i.e., materials which are uniform in all directions. Second, we will only consider materials that have no permanent electric or magnetic polarization.[i]

3.1 Dielectrics in Electric Fields

Somehow or another, dielectrics inside a capacitor are able to dramatically increase the amount of charge that can be stored and *decrease* the voltage across the capacitor. The usual explanation is that the dielectric itself partly charges, which both increases the amount of charge stored and decreases the net voltage. How does this work? In order to understand what is really going on, we have to think a bit about the microscopic nature of the dielectric.

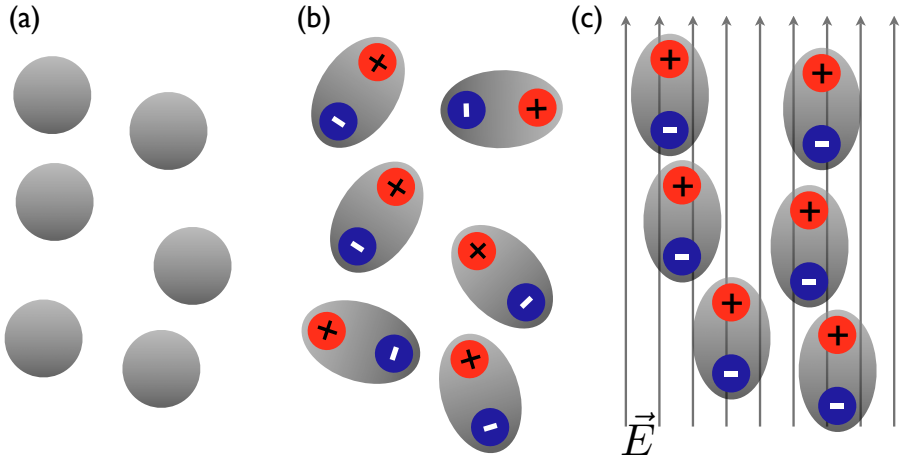

Figure 3.1: **(a)** *Atoms and many nuclei have no net charge separation without an electric field present.* **(b)** *Some "polar" molecules have a permanent electric dipole moment. Usually, these moments are oriented randomly from molecule to molecule, and the net moment is zero.* **(c)** *In an electric field, non-polar molecules can have an induced dipole moment, due to electrons and nuclei wanting to move in opposite directions in response to the field. Permanent dipoles remain bound, but can move or rotate slightly to align with the electric field. Either way, an overall dipole moment results.*

The dielectric itself contains a large number of atomic nuclei and electrons, but overall there are equal numbers of positive nuclei and electrons to make the dielectric overall neutral. We have said that charges in insulators are not mobile, so electrons and nuclei remain bound. What, then,

[i] Basically, no permanent magnets, ferroelectrics, or permanent dipole moments.

are the induced charges in the dielectric? Despite being bound, both electrons and nuclei in a dielectric can move very slightly without breaking their bonds. Electrons will attempt to move in the direction opposite the electric field between the plates, and nuclei will attempt to move in the opposite direction. As a result, tiny dipoles are formed inside the dielectric, which will be aligned along the direction of the electric field (see Figure 3.1). Random thermal motion of the atoms or molecules will limit the degree of alignment to an extent. In most materials the degree of alignment and the induced dipole strength are directly proportional to the external electric field. Essentially, an electric field induces a charge separation within the atom or molecule.

Some molecules have a natural charge separation or dipole moment already built in, so-called *polar* molecules such as water or CO_2. In these kinds of dielectrics, the built-in dipole moments are usually randomly aligned, and cancel each other out overall. An electric field exerts a torque on the dipoles, which tries to orients them along the electric field. Once again, random thermal motion works against this alignment, but the overall effect of the electric field is a net alignment, the degree of which is proportional to the applied electric field. Thus, in both polar and non-polar dielectrics, there is a net orientation of dipoles when an electric field is applied. The net dipole strength is far stronger in polar materials, and in the rest of the discussion below we will assume that our dielectric is made of polar molecules.

Now, what happens when we place our dielectric between two conducting plates? With no voltage applied between the plates, there is no electric field, and the tiny dipoles are randomly oriented, Fig. 3.2a. Once a voltage is applied to the plates, a constant electric is created between them, which serves to align the dipoles, Fig. 3.2b. The net alignment of dipoles within a dielectric leads to the surfaces of the dielectric being slightly charged, Fig. 3.2c. Within the bulk of the dielectric, dipoles will be aligned head-to-tail, and their electric fields will mostly cancel (Fig. 3.2b). At the surfaces of the dielectric, however, there will be an excess of positive charge on one side, and an excess of negative charge on the other. In this situation, the dielectric is said to be *polarized*. The dielectric is still electrically neutral on the whole, an equal number of positive and negative charges still exist, they have only separated due to the applied electric field.

These surface charges from the aligned dipoles look just like sheets of charge, in fact. This is the origin of our earlier statement that the dielectric picks up an induced charge on its surface – the part of the dielectric near the positive plate *does* build up a partial *negative* charge, and the part near the negative plate *does* build up a partial *positive* charge. What we missed in our initial analysis was the fact that in reality we are *aligning charges throughout the dielectric, even though only the surfaces have a net charge.* Not only are we storing energy in the surface charges, we are also storing energy by creating the aligned configuration of the dipoles! It took energy to orient them, so keeping them aligned is also storing energy for later release. In a sense, we actually store energy in the whole volume of the dielectric, not just at the surfaces.

The electric field due to these effective sheets of charge is *opposite* that of the applied electric field, and thus the total electric field – the sum of the applied and induced field – is smaller than if there were no dielectric. Thus, the dielectric reduces both the applied voltage and the electric field. The electric field due to the oriented dipoles inside the dielectric is usually proportional to the *total*

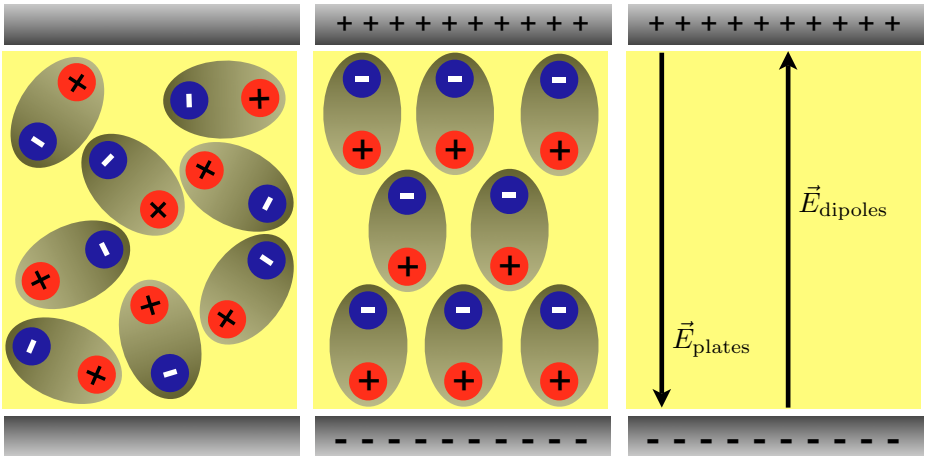

Figure 3.2: **(a)** *When no voltage is applied between the plates, the polar molecules align randomly, and there is no net dipole moment.* **(b)** *A voltage applied across the plates creates an electric field, which aligns the molecules.* **(c)** *The electric field from the voltage applied across the plates is partially canceled by the field due to the aligned dipoles.*

electric field they experience:

$$\mathsf{E}_{\text{dipoles}} = \chi_{\mathsf{E}}\,\mathsf{E}_{\text{total}} \tag{3.1}$$

where the constant of proportionality χ_{E} is called the *electric susceptibility*. It represents the relative strength of the dipoles within the material, or more accurately, how easily a material polarizes in response to an electric field. The total electric field the dipoles experience is not just the field due to voltage applied across the plates, but must also include *the field of all the other dipoles* as well:

$$\mathsf{E}_{\text{total}} = \mathsf{E}_{\text{plates}} - \mathsf{E}_{\text{dipoles}} \tag{3.2}$$

$$\mathsf{E}_{\text{total}} = \mathsf{E}_{\text{plates}} - \chi_{\mathsf{E}}\,\mathsf{E}_{\text{total}} \tag{3.3}$$

$$(1+\chi_{\mathsf{E}})\mathsf{E}_{\text{total}} = \mathsf{E}_{\text{plates}} \tag{3.4}$$

$$\implies \mathsf{E}_{\text{total}} = \frac{1}{1+\chi_{\mathsf{E}}}\mathsf{E}_{\text{plates}} \tag{3.5}$$

Thus, the field between the plates is *reduced* by a factor $1+\chi_{\mathsf{E}}$ by the presence of the dielectric (χ_{E} is always positive). We already know that for a parallel plate capacitor, $\Delta\mathsf{V} = \mathsf{E}\mathsf{d}$, where d is the spacing between the plates, so we can also readily find the effect of the dielectric on the potential difference between the plates:

$$\Delta\mathsf{V}_{\text{total}} = \frac{1}{1+\chi_{\mathsf{E}}}\mathsf{E}_{\text{plates}}\mathsf{d} = \frac{1}{1+\chi_{\mathsf{E}}}\Delta\mathsf{V}_0 = \frac{\Delta\mathsf{V}_0}{\epsilon_{\mathsf{r}}} \tag{3.6}$$

here we again use $\Delta\mathsf{V}_0$ for the voltage on the plates without the dielectric. We can go further and calculate the capacitance:

$$\mathsf{C} = (1+\chi_{\mathsf{E}})\epsilon_0\frac{\mathsf{A}}{\mathsf{d}} = \epsilon_{\mathsf{r}}\epsilon_0\frac{\mathsf{A}}{\mathsf{d}} = \kappa\mathsf{C}_0 \tag{3.7}$$

where C_0 is the capacitance without the dielectric present. Thus, our "relative dielectric constant" ϵ_r is simply related to the dielectric susceptibility χ_E, the ability of the dielectric to polarize in response to an electric field: $\epsilon_r = 1 + \chi_E$. This makes sense in a way – the more easily polarized the dielectric, the more easily it affects the capacitance. Also, since $\epsilon_r = 1$ for vacuum, $\chi_E = 0$, which also makes sense as the vacuum is not polarizable under normal conditions. Now we have a plausible *microscopic* origin for the effect of dielectrics in capacitors, and we know *why* the electric field and voltage are reduced, and the capacitance increased.

3.2 Maxwell's equations in linear media

The problem of interest is to figure out how fields behave once they enter matter. In the *static* example above, the total electric field inside a material was a factor $\epsilon_r = 1 + \chi_E$ smaller than the external field. If we think about this from another viewpoint, one could just as easily say that an external field (E_{plates} in the previous example) is *larger* than the field inside the material (E_{total}) by a factor ϵ_r:

$$E_{external} = (1 + \chi_E) E_{internal} = \epsilon_r E_{internal} \tag{3.8}$$

This "external" field is usually known as the *electric displacement field* $\vec{D}$. Of course what we call it makes absolutely no difference. For mostly historical reasons, however, it has an extra factor ϵ_o:

$$\vec{D} = \epsilon_o E_{external} = \epsilon_o (1 + \chi_E) \vec{E} = \epsilon \vec{E} \tag{3.9}$$

The equation above still just tells us only that the field inside a dielectric is a factor ϵ_r lower than it is outside. Now, we can do *exactly the same thing* for the magnetic field, and define an auxiliary magnetic field $\vec{H}$ which takes into account static magnetic polarization. The main difference is now that the auxiliary field $\vec{H}$ is the "external" field and $\vec{B}$ the "internal" field:

$$\vec{B} = \mu_o H_{external} = \mu_o (1 + \chi_M) \vec{H} = \mu \vec{H} \tag{3.10}$$

We will deal exclusively with non-magnetic materials, for which $\mu_r \approx 1$, so $\vec{B} \approx \mu_o \vec{H}$, so we won't need to worry about the $\vec{H}$ field very much.

The sort of materials we have just considered are known as *linear, homogeneous, isotropic* materials - the internal field is linearly proportional to the external field, and the material's properties are homogeneous and independent of direction. All we need to do now is put the modified fields into Maxwell's equations, which mostly amounts to replacing ϵ_o with ϵ and μ_o with μ:

$$\nabla \cdot \vec{E} = \frac{\rho_{\text{free}}}{\epsilon} \tag{3.11}$$

$$\nabla \cdot \vec{B} = 0 \tag{3.12}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3.13}$$

$$\nabla \times \vec{B} = \mu \vec{j}_{\text{free}} + \mu\epsilon \frac{\partial \vec{E}}{\partial t} \tag{3.14}$$

Here ρ_{free} means that we consider only the free, unbound charges, and j_f includes only free, unbound currents. What are free charges? Simply, ones that can leave, which excludes charges which are part of permanent or induced dipoles like those in the previous section. What about free currents? Same idea, these are simply currents made up of free charges, which means they are the ordinary currents you are used to thinking about, not microscopic current loops giving rise to magnetic moments.

The modifications required to Maxwell's equations are relatively straightforward for linear, isotropic, homogenous media. The problem comes from the fact that in general, μ and ϵ are frequency-dependent, and our larger task is finding reasonable models for $\epsilon(\omega)$ and $\mu(\omega)$.

3.2.1 Electromagnetic waves in linear media

We will make one further assumption before moving on. If we have a material with free charges, then it is possible for a free current to flow. We will assume that Ohm's law remains valid when there is free charge (which essentially amounts to assuming that charges undergo collisions frequently compared to the frequency of radiation, and that the fields aren't too large), such that

$$\vec{j}_{\text{free}} = \sigma \vec{E} \tag{3.15}$$

Since the material is already presumed to be isotropic, σ need not depend on spatial coordinates, though it may still be frequency-dependent. We can now re-write Ampere's law in linear, "Ohmic" media as

$$\nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon \frac{\partial \vec{E}}{\partial t} \tag{3.16}$$

What are the simplest solutions of Maxwell's equations in linear media? Let us presume we have an electric field present, and we will define the x axis to be in the direction of this field, such that $\vec{E} = E_x \hat{x}$. This implies that the $\vec{B}$ field has only a $\hat{y}$ component, consistent with the right-hand rule:[ii]

$$\nabla \times \vec{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \frac{\partial E_x}{\partial z}\hat{y} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial B_y}{\partial t}\hat{y} \tag{3.17}$$

[ii] Here we use the shorthand $\partial_x \equiv \frac{\partial}{\partial x}$.

Likewise, we can show

$$\nabla \times \vec{B} = -\frac{\partial B_y}{\partial z}\hat{x} = \left(\mu\sigma E_x + \mu\epsilon\frac{\partial E_x}{\partial t}\right)\hat{x} \tag{3.18}$$

So far, the fields E_x and B_y are perfectly general. A field of specific utility would be a sinusoidal electromagnetic (EM) wave. A sinusoidal EM wave with components E_x and B_y must be traveling along the $\hat{z}$ direction, according to

$$E_x = E_{xo}e^{i(kz-\omega t)} \tag{3.19}$$

$$B_y = B_{yo}e^{i(kz-\omega t)} \tag{3.20}$$

For this simplest EM wave, we know that the propagation speed of the wave is $v = \omega/k$, which in vacuum gives $v = c$. Using these as our fields, we can re-write Eqs. 3.17 and 3.18 as:

$$ikE_{xo}e^{i(kz-\omega t)} = i\omega B_{yo}e^{i(kz-\omega t)} \tag{3.21}$$

$$(\mu\sigma - i\omega\mu\epsilon)\,E_{xo}e^{i(kz-\omega t)} = -ikB_{yo}e^{i(kz-\omega t)} \tag{3.22}$$

Canceling the complex exponential terms, we are left with:

$$ikE_{xo} - i\omega B_{yo} = 0 \tag{3.23}$$

$$(\mu\sigma - i\omega\mu\epsilon)\,E_{xo} + ikB_{yo} = 0 \tag{3.24}$$

If you solve the first equation for B_{yo} and plug it into the second, you find a condition relating k and ω:

$$k^2 = \omega^2\mu\epsilon + i\omega\mu\sigma \tag{3.25}$$

To be sure, this is not as simple as our previous relationship in free space, which simply gave $c = \omega/k$. Evidently, the wave velocity is no longer so simply determined in real matter. What we have is now is a *dispersion relationship* between k and ω, which means that while the wave speed is still ω/k, it is no longer a constant but explicitly dependent on the frequency of the electromagnetic radiation involved. To understand better what this all means, we should first look at some limiting cases.

3.2.1.1 Limiting case: Insulators

In a purely insulating material, there is no free charge, and hence no conductivity: $\sigma = 0$. In this case, our dispersion relationship reads:

$$k = \omega\sqrt{\mu\epsilon} \tag{3.26}$$

In this case, since μ and ϵ are positive real numbers, k is real, and the wave solution is still just a propagating wave. However, using $\mu = \mu_r\mu_o$ and $\epsilon = \epsilon_r\epsilon_o$, we notice something else:[iii]

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_o\epsilon_o\mu_r\epsilon_r}} = \frac{c}{\sqrt{\mu_r\epsilon_r}} \equiv \frac{c}{n} \tag{3.27}$$

Here we have an interesting result: the wave propagation speed is no longer c, it is a factor $n = \sqrt{\mu_r\epsilon_r}$ *lower* in the medium! Evidently, when light enters a non-conducting medium, its propagation velocity is reduced. The amount of reduction n is the *index of refraction* of the substance, and depends only on the relative dielectric constant $\epsilon_r = \epsilon/\epsilon_o$ and permeability $\mu_r = \mu/\mu_o$. For non-magnetic media, where $\mu_r \approx 1$, we have the simple result $n \approx \sqrt{\epsilon_r}$. The better the dielectric, the more the propagation of light is slowed. The implications of this fact are *refraction*, the bending of light when passing between different media.

3.2.1.2 Limiting case: Conductors

In a purely conducting material, we have no bound charges. In fact, since ideal conductors can cancel out external electric fields perfectly, they must have $\chi_E = -1$, meaning $\epsilon = (1 + \chi_E)\epsilon_o = 0$. Basically: since ideal conductors can't support regions of isolated charge, they have no dielectric constant. This simplifies our dispersion relationship:

$$k = \sqrt{i\omega\mu\sigma} \tag{3.28}$$

Now this is odd: k is now complex? The meaning of this is simple enough to realize if we just plug a complex value for k into our original field. Let $k = k_r + ik_i$, then

$$E_x = E_{xo}e^{i(k_r z + ik_i z - \omega t)} = E_{xo}e^{i(k_r z - \omega t)}e^{-k_i z} \tag{3.29}$$

An complex k just means that we have an oscillating field which is *also* exponentially damped, with a length scale $1/k_i$. A purely real k corresponds to a propagating wave, while a purely imaginary k corresponds to an evanescent exponentially-decaying amplitude. That k is complex is then perfectly sensible for a conductor – an incident electric field will be rapidly damped and can't penetrate the conductor, as we would expect. How rapidly will the EM wave be damped? Noting that $\sqrt{i} = (1+i)/\sqrt{2}$, we can find k_r and k_i from Eq. 3.28:

$$k_r = k_i = \sqrt{\frac{1}{2}\omega\mu\sigma} \tag{3.30}$$

Thus, an incident EM wave impinging on a conductor decays with a characteristic length scale

[iii] That is, if you remember that $c = 1/\sqrt{\mu_o\epsilon_o}$.

$$\frac{1}{k_i} = \delta = \sqrt{\frac{2}{\omega\mu\sigma}} \tag{3.31}$$

This length scale δ is also known as the *skin depth*. The higher the frequency or conductivity, the less the EM wave penetrates into the conductor, while at relatively low frequencies (say, those in typical ac circuits) the penetration depth is significantly larger. This means that, among other things, alternating currents at low frequencies can spread evenly through the diameter of a conductor, while at higher frequencies, the EM waves can only propagate significantly within a few δ's of the surface. For Cu, the skin depth at 60 Hz is about 8 mm, while at 1 MHz it is 60 μm, and at 1 GHz, it is only about 2 μm. Of course, if this is true of EM waves, it must also be true of the currents that result from them, since we have assumed $\vec{j} = \sigma\vec{E}$ – just like the field, the current density in a conductor decays exponentially from its surface (which is why we assume thin wires for most problems!). Thus, high frequency cables should be made out of many braided smaller wires, rather than a single large conductor that might suffice at low frequencies. Anything much larger than a radius of ~8 mm at 60 Hz is essentially wasting copper, since the central portion won't be carrying much current anyway!

Perhaps more importantly for the present discussion, we know something general about how EM waves interact with conductors. If the wave can't be transmitted, and it can only be absorbed in a tiny region a few δ's from the surface, the bulk of the wave must be reflected. Conductors reflect EM waves, so metals are shiny!

3.2.1.3 General case

Returning to the general case, we have

$$k = \sqrt{\omega^2\mu\epsilon + i\omega\mu\sigma} = k_r + ik_i \tag{3.32}$$

If k is purely real at frequency ω, we have wave propagation, and the material is transparent at that frequency. This will only happen if $\sigma = 0$, for a pure insulator. If k is purely imaginary at frequency ω, we have attenuation, and unless the material is very thin (thickness of order δ), it will be completely opaque at frequency ω. This will only happen in a pure conductor, when $\epsilon = 0$. One can also find k_r and k_i explicitly, though it is messy:

$$k_r = \sqrt{\frac{\mu}{2}}\sqrt{\omega\sqrt{\epsilon^2\omega^2 + \sigma^2} + \omega^2\epsilon} \tag{3.33}$$

$$k_i = \sqrt{\frac{\mu}{2}}\sqrt{\omega\sqrt{\epsilon^2\omega^2 + \sigma^2} - \omega^2\epsilon} \tag{3.34}$$

One can see in the general case that it is still true that if $\sigma = 0$, there is no imaginary part of k, and EM waves propagate in the medium. Many real materials are neither pure conductors or pure insulators, and will display a bit of each behavior. Roughly speaking, at a given frequency,

a higher conductivity leads to more attenuation, while a higher dielectric constant leads to more propagation.

3.3 Microscopic Picture of Insulators and Conductors

The "linear media" approach is extremely valuable, and already points us toward the laws of reflection and refraction, it comes with no microscopic explanation. A true microscopic explanation requires a quantum-mechanical analysis. Nonetheless, our model of atoms as damped, oscillating charges can produce quite reasonable behavior, and gives a a bit more insight into how radiation and matter interact on a microscopic level.

We will start by considering a damped, oscillating charge driven by an external electric field, whose behavior was derived in Sec. 1.3.6. The position of the charge followed

$$x(t) = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \cos(\omega t + \varphi) \tag{3.35}$$

$$\tan\varphi = \frac{2\omega\omega_o\gamma}{\omega^2 - \omega_o^2} \tag{3.36}$$

We will first consider a collection of these oscillating charges in an insulating (dielectric) medium, and wee what more we can say about the dielectric behavior.

3.3.1 Motion of charges in (imperfect) dielectrics

If we are going to consider EM waves in real materials, then our oscillating charge is almost invariably an electron bound to a positively charged and (essentially) fixed atomic nucleus. Our oscillating electron bound to the nucleus thus constitutes an electric dipole. If the position of the charge relative to the nucleus is given by $x(t)$, the dipole moment is then $p = ex(t)$. Using the form of $x(t)$ above, each oscillating electron contributes a dipole moment

$$\vec{p} = e\vec{x}(t) = \frac{e^2\vec{E}}{m(\omega_o^2 - \omega^2 - i\omega\gamma)} \equiv \alpha\epsilon_o\vec{E} \tag{3.37}$$

The quantity α is known as the *atomic polarizability*, defined as

$$\alpha = \frac{e^2}{m\epsilon_o}\left(\frac{1}{\omega_o^2 - \omega^2 - i\omega\gamma}\right) \tag{3.38}$$

Thus, the charges within a medium give rise to a polarization due to the oscillatory motion induced by an incident EM wave. This polarization is *dynamic*, as it occurs only when oscillating electric fields are present, and is in *addition to* the static polarization expected in a dielectric. If we have N atoms in our medium, and presume that they oscillate incoherently (see the beginning of Sect. 1.3.7), then the total polarization due to the induced oscillations of charges in the medium

is just $N\vec{p}$, so the total field in the medium must be the external field, plus the static polarization due to the electric susceptibility χ_E, plus the dynamical effects:

$$\vec{D} = \epsilon_o (1+\chi_E)\vec{E} + N\vec{p} = \epsilon_o (1+\chi_E)\vec{E} + N\alpha\epsilon_o\vec{E} \equiv \epsilon\vec{E} \tag{3.39}$$

Here ϵ is now the effective dielectric constant including both the static and dynamic effects, and evidently,

$$\epsilon(\omega) = \epsilon_o (1+\chi_E) + N\epsilon_o\alpha(\omega) = \epsilon_o (1+\chi_E) + \frac{Ne^2}{m}\left(\frac{1}{\omega_o^2 - \omega^2 - i\gamma\omega}\right) \tag{3.40}$$

What is interesting is that including the dynamical effects, the driven and damped motion of the individual charges, we now have an ϵ that can be *complex*, and which is explicitly frequency-dependent. What does a complex ϵ mean? Recall that for a non-conductor ($\sigma = 0$), the wave velocity is given by

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{\omega}{k} \qquad \text{or} \qquad k = \frac{\omega}{\sqrt{\mu\epsilon}} \tag{3.41}$$

For non-magnetic materials, $\mu \approx \mu_o$, so a complex ϵ means that k must be complex since ω must be real. As discussed above for conductors, a purely real k means propagating waves, a purely imaginary k means attenuated or absorbed waves, and a complex k means a mix of the two, propagation with attenuation. If ϵ has an imaginary part, that means that the dielectric material is "lossy," and perfect propagation of EM waves is no longer the case. We can explicitly find the real and imaginary parts of ϵ to see what conditions lead to lossy behavior:

$$\Re(\epsilon) \equiv \epsilon_1 = \epsilon_o (1+\chi_E) + \frac{Ne^2}{m}\frac{\omega_o^2 - \omega^2}{(\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{3.42}$$

$$\Im(\epsilon) \equiv \epsilon_2 = \frac{Ne^2}{m}\frac{\gamma\omega}{(\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{3.43}$$

We can already see that the larger the damping, the larger ϵ_2, the imaginary part of ϵ, and the larger the attenuation. Just as in the case of the conductor, the EM waves will be damped exponentially with a characteristic distance given by $1/k_i$:

$$k_r = \omega\sqrt{\mu\epsilon_1} \tag{3.44}$$

$$k_i = \omega\sqrt{\mu\epsilon_2} \tag{3.45}$$

Damping of the oscillations leads to attenuation of the EM wave, as it must: incident EM waves are absorbed and re-radiated by charges in the medium, but with a slightly smaller amplitude. Subsequent charges absorb the re-radiated EM waves and give back a bit smaller amplitude, and the EM wave is further and further attenuated as it travels further into the medium. If the medium

is very thick compared to $1/k_i$, there is effectively no transmission, so the wave is predominantly reflected with possibly some absorption at the surface. If the medium is thin, a portion of the incident intensity is transmitted.

Only with non-zero damping can we have pure propagation (transmission) of EM waves, but we know we will always have at least radiation damping to content with, so perfect transmission is impossible in this simple model (though since radiation damping is very small, it can be nearly perfect). We can also see a strong frequency dependence – ϵ_2 is zero as $\omega \to 0$, peaks strongly at resonance ($\omega = \omega_o$), but is finite above resonance. This is analogous to the result we found in deriving the scattering cross section: both scattering and attenuation are more effective above resonance.

One more thing we can investigate is the behavior of a parallel-plate capacitor with a complex dielectric inside. For a capacitor with plates of area A and spacing d, the impedance would be

$$Z^{-1} = i\omega C = i\omega \frac{\epsilon A}{d} = i\omega \frac{\epsilon_1 A}{d} - \omega \frac{\epsilon_2 A}{d} \tag{3.46}$$

The impedance has both real and imaginary parts! If we interpret $-\omega\epsilon_2$ as a conductivity, $\sigma = -\omega\epsilon_2$, this is just a resistor and capacitor in parallel – our imperfect dielectric is a "leaky" capacitor, or an ideal capacitor with a parallel resistor shorting out the two plates and letting some charge leak away. This effective parallel RC circuit then has a time constant $\tau = \epsilon_1/\omega\epsilon_2$, *independent of geometry.*

3.3.2 Motion of Charges in (imperfect) Conductors

If we wish to consider a microscopic picture of conductors, we need to add some new physics. First, in an ideal conductor the electrons are free, any *any* electric field will lead to displacement. This means that the resonance frequency is effectively zero, $\omega_o = 0$. Second, the fact that both conductors and lossy dielectrics exhibit EM wave attenuation means that the distinction between a "lossy" dielectric and a conductor is essentially artificial: saying a material has a non-zero imaginary dielectric constant ϵ_2 amounts to saying that the material has non-zero conductivity.

To see the last point more clearly, we can notice the two expressions we had for $k(\omega)$: the first coming from our macroscopic approach, the second from our microscopic "dielectric-centric" view:

$$k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega = \mu\epsilon_1\omega^2 + i\mu\epsilon_2\omega^2 \tag{3.47}$$

This tells us that the *an imaginary dielectric constant is just another view of conductivity.* If we wish to take the "conductor-centric" viewpoint, we can simply identify the complex frequency-dependent dielectric function with a complex conductivity (with $\omega_o = 0$), and retain the static frequency-independent dielectric constant. We need only make the identification from the dispersion relationship above that $\sigma = \omega\epsilon/i$:

$$\sigma(\omega) = \frac{\omega\epsilon}{i} = \frac{\omega N e^2}{im}\left(\frac{1}{-\omega^2 - i\gamma\omega}\right) = \frac{Ne^2}{\gamma m}\left(\frac{1}{1 - i\omega/\gamma}\right) \equiv \frac{\sigma_o}{1 - i\omega/\gamma} \tag{3.48}$$

$$\epsilon = \epsilon_o\,(1 + \chi_E) \tag{3.49}$$

Here $\sigma_o = Ne^2/\gamma m$ is a real constant (for a given material) with dimensions of conductivity. If we compare this with our derivation of Ohm's law, where we found $\sigma = Ne^2\tau/m$ in the static case, we come to an interesting conclusion: $\gamma = 1/\tau$, and the damping constant is just the inverse of the collision time! In a conductor, physically the damping is due to collisions, which fritter away energy from the EM wave in the form of heat.

Thus, our microscopic picture provides two equivalent viewpoints. For an imperfect dielectric, we can use a complex dielectric function, the imaginary part of which leads to wave attenuation and the real part of which leads to wave propagation. For an imperfect conductor, we can use a complex conductivity, the real part of which leads to wave attenuation and the complex part of which leads to wave propagation. If we like, we can add a constant dielectric constant to the conductor or a constant conductivity contribution to the insulator so the conductor has a static electric polarization at zero frequency or the dielectric has a non-zero conductivity at zero frequency. Which viewpoint is used is a matter of taste – say, whether one considers the material in question to be more like a conductor or an insulator – but the essential physics is the same. At high enogh frequencies, real materials have both a dielectric and conductive response.

We can explicitly write the real and complex parts of the conductivity and draw some conclusions about the behavior of conductors at various frequencies:

$$\Re(\sigma) \equiv \sigma_1 = \frac{\sigma_o}{1 + \omega^2\tau^2} \tag{3.50}$$

$$\Im(\sigma) \equiv \sigma_2 = \frac{\sigma_o\omega\tau}{1 + \omega^2\tau^2} \tag{3.51}$$

The first conclusion to take away is that at low frequencies, the imaginary conductivity tends to zero and the real conductivity toward the Ohmic result σ_o – the material behaves as a conductor. Second, at high enough frequencies we can't neglect the dielectric behavior of a conductor, it starts to behave a bit as an insulator. Lastly, if we use this conductivity to find the impedance of a conductor of cross-sectional area A and length l, we find (ignoring the static dielectric constant)

$$Z^{-1} = \frac{\sigma(\omega)A}{l} = \frac{\sigma_1 A}{l} + i\frac{\sigma_2 A}{l} = R_{\text{eff}}^{-1} + i\omega C_{\text{eff}} \tag{3.52}$$

Again, the impedance has both real and imaginary parts in parallel, if we identify $\sigma_2/\omega = \epsilon$ as a dielectric constant, similar to the way we identified $\omega\epsilon_2$ as a conductivity for an imperfect dielectric. The real part of the conductivity behaves like a normal resistor, the imaginary like a capacitor. Just like our imperfect dielectric, an imperfect conductor behaves as a resistor and capacitor in parallel, with time constant $\sigma_2/\sigma_1 = \omega\tau$. The presence of the static dielectric constant will of course add to

the overall capacitance.

Since we know now the real part of the conductivity is what contributes to normal current flow, and the imaginary part to charge accumulation and an effective capacitance, we can derive a figure of merit for how good our conductor is, the ratio of the imaginary to real conductivity $\sigma_2/\sigma_1 = \omega\tau$. Note that $\omega\tau$ is dimensionless, and is essentially the ratio of the field frequency to the collision frequency. For $\omega\tau \gg 1$, the collision time is short compared to the period of the EM waves, and we have a good conductor. For $\omega\tau \ll 1$, EM wave's period is shorter than the time between collisions, and we have a material behaving more like a dielectric. If $\omega\tau \sim 1$, the material is simultaneously a bad dielectric and a bad conductor.

3.3.3 Mixed conductivity and dielectric behavior

So far, we have analyzed a material with complex conductivity, a material with a complex dielectric constant, and found the two approaches to be equivalent. We can also consider the material to have both a real conductivity and dielectric constant, with neither being imaginary, and the physics is the same. In that case, we simply say that the dielectric constant is ϵ_1 as derived in our "dielectric-centric" model, and the conductivity is σ_1 as derived in the "conductor-centric" model.

$$\sigma(\omega) = \frac{\sigma_o}{1 + \omega^2\tau^2} \tag{3.53}$$

$$\epsilon(\omega) = \epsilon_o\,(1 + \chi_E) + \frac{Ne^2}{m}\frac{\omega_o^2 - \omega^2}{(\omega_o^2 - \omega^2)^2 + \gamma^2\omega^2} \tag{3.54}$$

$$k^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{3.55}$$

This approach simply assumes that the material has mixed conductivity and dielectric behavior, which one can still view as a "leaky" dielectric or a "lossy" conductor. In either case, the macroscopic behavior is still equivalent to a parallel RC circuit, and we still have wave propagation and attenuation.

Using this more agnostic viewpoint, we can go one step further and apply the charge continuity equation to see how charge density behaves in an imperfect conductor. Noting that σ is still assumed to be spatially isotropic,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\vec{j} = -\nabla\cdot\left(\sigma\vec{E}\right) = -\sigma\nabla\cdot\vec{E} \tag{3.56}$$

We also $\nabla\cdot\vec{E} = \rho/\epsilon$, so

$$\frac{\partial\rho}{\partial t} = -\frac{\sigma\rho}{\epsilon} \tag{3.57}$$

$$\Longrightarrow \qquad \rho(t) = \rho_o e^{-(\sigma/\epsilon)t} \tag{3.58}$$

Here ρ_o is the charge density at time $t=0$. What we can see is that any charge deposited on our mixed medium leaks away exponentially with a time constant $\tau=\epsilon/\sigma=\sigma_2/\sigma_1=\epsilon_1/\omega\epsilon_2$, just as we noted from looking at the impedance of resistors and capacitors with imperfect conductors and dielectrics, respectively. The better the conductivity, the faster the charge leaks away, the better the dielectric, the slower it leaks away.

3.4 EM waves in conductors

Within the conductor-centric view, with a complex conductivity $\sigma(\omega)=\sigma_1(\omega)+i\sigma_2(\omega)$ and real, static dielectric constant ϵ, we can write the dispersion relationship as

$$k^2=\omega^2\mu\epsilon+i\omega\mu\sigma(\omega)=\omega^2\mu\epsilon+i\omega\mu\left(\frac{\sigma_o}{1-i\omega\tau}\right)=\omega^2\mu\epsilon\left(1+\frac{i\sigma_o}{\omega\epsilon\,(1-i\omega\tau)}\right) \tag{3.59}$$

At low frequencies ($\omega\tau\ll 1$), the EM waves have a low frequency compared to the collision frequency. We may then neglect the $i\omega\tau$ term in the denominator, and the above simplifies to our previous result:

$$k^2=\omega^2\mu\epsilon+i\omega\mu\sigma_o \tag{3.60}$$

On the other hand, for high frequencies ($\omega\tau\gg 1$), the EM waves have a higher frequency than the collisions, and we can expect qualitatively different behavior. In this regime, the $i\omega\tau$ term in the denominator dominates:

$$k^2=\omega\mu\epsilon\left(1-\frac{\sigma_o}{\omega^2\epsilon\tau}\right)=\omega\mu\epsilon\left(1-\frac{\omega_p^2}{\omega^2}\right) \tag{3.61}$$

The quantity $\omega_p^2=\sigma_o/\epsilon\tau$ has units of angular frequency, and is known as the *plasma frequency*. Below this frequency, the term in brackets is negative, k is imaginary, and the EM wave is reflected (provided the conductor is thicker than a few δ's). What we can notice now is that if $\omega>\omega_p$, k is purely real – there is *no attenuation* of the EM wave! For frequencies above ω_p, the conductor is transparent to incident radiation, while below ω_p it is opaque. Roughly, below the plasma frequency the material behaves like a conductor, while above it behaves more like a dielectric, and the plasma frequency is the frequency at which the charges can no longer keep up with the rapid oscillations of the EM field. Indeed, in the limit $\omega\ll\omega_p$ we recover the result $\omega/k=1/\sqrt{\mu\epsilon}=n/c$.

This particular result explains why many common metals are transparent to deep UV light. For instance, if we take $\epsilon\sim\epsilon_o$, $n\sim 6\times 10^{28}\,m^{-3}$, reasonable values for a good conductor like Cu or Al, we find a plasma wavelength $\lambda_p=2\pi/\omega_p\sim 140\,nm$. In particular, the alkali metals are transparent in the near UV, the table below lists a few interesting cases

element	λ_p
Cs	440 nm
Na	210 nm
Li	205 nm
Al	200 nm

3.4.1 Reflection

Knowing that light is nothing more than EM waves, and that conductors thicker than a few skin depths will reflect EM waves with frequencies below their plasma frequency, we are now in position to derive the law of reflection. All we really need are the boundary conditions on the electric field and the properties of conductors.

Say we have an EM wave with $\omega < \omega_p$ incident on a metal-vacuum interface at angle θ_i with respect to the interface normal. We know that the EM wave will be reflected so long as the conductor is thick compared to δ. We also know that a conductor can't have any surface charge, so the component of the electric field perpendicular to the interface cannot change upon reflection:

$$E_{\perp,i} - E_{\perp,r} = \sigma/\epsilon = 0 \qquad \Longrightarrow \qquad E_{\perp,i} = E_{\perp,r} \tag{3.62}$$

Thus, the incident and reflected waves have the same perpendicular components of the electric field. On the other hand, we know that at any interface, the parallel component of the electric field must be conserved, so the incident and reflected EM waves also have the same parallel component:

$$E_{||,i} - E_{||,r} = 0 \qquad \Longrightarrow \qquad E_{||,i} = -E_{||,r} \tag{3.63}$$

This implies that the incident and reflected EM waves make the same angle with respect to the surface normal:

$$\tan\theta_i = \frac{E_{\perp,i}}{E_{\perp,i}} = -\frac{E_{||,i}}{E_{||,i}} = -\tan\theta_i \tag{3.64}$$

Thus, $\theta_i = -\theta_r$, the angle of incidence is equal to the angle of reflection. This is all that is required, along with the geometry of the conductor, to develop reflective optics, i.e., mirrors!

3.5 EM waves in insulators

In insulating materials with negligible ϵ_2 or σ, we have already discovered that EM waves travel at reduced speed $v = c/n$, where $n = \sqrt{\mu_r \epsilon_r}$ is the index of refraction. If we consider non-magnetic materials ($\mu \approx \mu_o$, we have already found

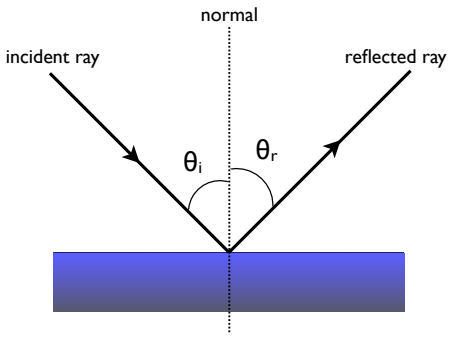

Figure 3.3: *An EM wavefront (or "ray") incident on a conductor is reflected at the same angle at which it is incident.*

$$n = \sqrt{\epsilon_r} = \sqrt{1 + \chi_E + \frac{Ne^2}{m}\left(\frac{1}{\omega_o^2 - \omega^2}\right)} \tag{3.65}$$

For most common insulators, $\omega_o < \omega$ for visible light (ω_o is in the UV, typically), so n is purely real, and we have essentially pure propagation, albeit at reduced velocity. For visible light, higher frequency blue light has a higher index of refraction than lower frequency red light, and thus blue light is more strongly slowed by most dielectric media (so long as ω_o above visible frequencies).

3.5.1 Propagation delay

One implication of the reduced speed is a propagation delay for EM waves traveling through a dielectric medium. Imagine that we have a non-absorbing glass plate of index n and thickness Δy, which stands between a source S and observer P. Without any media present, to cross a distance Δy at speed c the wave will take a time

$$t_{wo} = \frac{\Delta y}{c} \tag{3.66}$$

With the medium present, the wave has a reduced speed of $v = c/n$, and the time required is

$$t_w = \frac{n\Delta y}{c} \tag{3.67}$$

Thus, the presence of the media delays the wave by an amount

$$\delta t = t_w - t_{wo} = \frac{\Delta y}{c}(n-1) \tag{3.68}$$

This time delay shows up as an extra phase compared to the original unimpeded wave. That is, since the wave in the presence of the glass is delayed by a time δt compared to the unimpeded wave, we must shift the time coordinate back by that amount. This amounts to making the substitution $t \to t - \delta t$ in the original wave to account for the time lag, or equivalently, multiplying by a phase factor $e^{-i\omega\delta t}$

$$E_p = E_u(t-\delta t) = E_o e^{i\omega(t-\delta t-y/c)} = E_o e^{i\omega(t-(n-1)\Delta y/c-y/c)} \tag{3.69}$$

$$= E_o e^{i\omega(t-y/c)} e^{-i\omega(n-1)\Delta y/c} = E_u e^{-i\omega\delta t} \tag{3.70}$$

The latter form makes it clear that the magnitude of E is unchanged by the presence of the glass, only the phase is altered. If the plate is thin or the index of refraction is not much different than 1 (its vacuum value), i.e. $(n-1)\Delta y \ll 1$, we may use the approximation $e^x \approx 1+x$ for the phase factor:

$$e^{-i\omega\delta t} = e^{-i\omega(n-1)\Delta y/c} \approx 1 - i\omega\,(n-1)\,\Delta y/c \tag{3.71}$$

Noting that $-i = e^{-i\pi/2}$,

$$e^{-i\omega\delta t} \approx 1 + \omega\,(n-1)\,\frac{\Delta y}{c} e^{-i\pi/2} \tag{3.72}$$

Substituting back into the equation for E_p,

$$E_p = E_u e^{-i\omega\delta t} = E_u\left(1 + \omega\,(n-1)\,\frac{\Delta y}{c} e^{-i\pi/2}\right) = E_u + \frac{\omega\,(n-1)\,\Delta y}{c} E_u e^{-i\pi/2} \tag{3.73}$$

The observer sees a wave which is the sum of two terms: the first is the wave that would be observed without the glass present, the second is a $\pi/2$ phase-shifted interference term, representing the out-of-phase field due to the oscillators in the glass plate.

3.5.2 Index of low density materials

If we have a material of low density, such as a gas, we may suppose that the second term under the radical in Eq. 3.65 is small compared to 1. That is,

$$\frac{Ne^2}{2\epsilon_o m_e\,(\omega_o^2 - \omega^2)} \ll 1 \tag{3.74}$$

This is valid at low enough density N, provided the frequency ω is not right at the resonance frequency ω_o. Specifically, let us imagine that we are dealing with 200 nm UV light, $\omega \sim 1\times10^{16}$ Hz. If we consider driving frequencies in the visible, no lower than 350 nm ($\omega_o \sim 5\times10^{15}$ Hz), then $(\omega_o^2-\omega^2)^{-1} \sim 10^{-32}$. Given that $e^2/\epsilon_o m_e \sim 3000$, our limit on density is

$$1 \gg \frac{Ne^2}{2\epsilon_o m_e \left(\omega_o^2 - \omega^2\right)} \tag{3.75}$$

$$N \ll \frac{2\epsilon_o m_e}{e^2}\left(\omega_o^2 - \omega^2\right) \sim 10^{35}\,\mathrm{m}^{-3} \tag{3.76}$$

Our atmosphere at sea level has $\sim 10^{25}$ molecules per m^3, so this is a well-justified approximation for gases at visible frequencies, so long as we are reasonably far from the resonance frequency. Given the inequality in Eq. 3.74, we may approximate the radical in Eq. 3.65 with $\sqrt{1+x} \approx 1 + x/2$ since $x \ll 1$. Thus,

$$n(\omega) = \sqrt{1 + \frac{Ne^2}{\epsilon_2 m_e}\frac{1}{\omega_o^2 - \omega^2}} \approx 1 + \frac{Ne^2}{2\epsilon_2 m_e}\frac{1}{\omega_o^2 - \omega^2} \tag{3.77}$$

From this we can see that for $\omega < \omega_o$, higher frequencies have a more strongly reduced speed (higher n) than lower frequencies, so in the visible blue light will be slowed more than red (just as blue light is scattered more than red by atmospheric gasses, as we found in considering the scattering cross section). As we shall see in the next section, this means that the direction of blue light incident at an oblique angle on a dielectric shows greater angular deviation than red light, which is one reason why at sunset the horizon near the sun looks red. In more dense media, as we noted above the same basic conclusion holds, which explains the characteristics of a prism.

3.5.3 The law of refraction

Fermat's principle and refraction, Snell's law

A

Evaluating $\int_0^\infty x^3\,dx/\,(e^x-1)$*

Pathologically, the best way to calculate the integral

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx \tag{A.1}$$

is to calculate a more general case and reduce it to the answer we require. Take the following integral

$$\int_0^\infty \frac{x^n}{e^x-1}\,dx = \int_0^\infty \frac{x^n e^{-x}}{1-e^{-x}}\,dx \tag{A.2}$$

The denominator is always less than one, and is in fact the sum of a geometric series with common multiplier e^{-x}:

$$\frac{1}{1-e^{-x}} = \sum_{k=0}^{\infty} e^{-kx} \tag{A.3}$$

If we substitute in this series, our integral becomes

$$\int_0^\infty x^n e^{-x} \sum_{k=0}^{\infty} e^{-kx}\,dx \tag{A.4}$$

We can bring the factor e^{-x} inside our summation, which only shifts the lower limit of the sum from 0 to 1, leaving:

$$\int_0^\infty x^n \sum_{k=1}^{\infty} e^{-kx}\,dx \tag{A.5}$$

Now make a change of variables $u=kx$, meaning

$$x^n = \frac{u^n}{k^n} \tag{A.6}$$

$$dx = \frac{du}{k} \tag{A.7}$$

With this change of variables, our integral is:

$$\int_0^\infty \frac{u^n}{k^n} \sum_{k=1}^{\infty} e^u \frac{du}{k} = \int_0^\infty u^n \sum_{k=1}^{\infty} e^u \frac{du}{k^{n+1}} \tag{A.8}$$

Each term in the sum represents an integral over u, all of which are convergent. This means we can interchange the order of summation and integration:

$$\sum_{k=1}^{\infty}\frac{1}{k+1}\int_0^\infty u^n e^{-u}\,du \tag{A.9}$$

The integral on the right side is the definition of the Gamma function $\Gamma(n+1)$, while the summation is then the definition of the Riemann zeta function $\zeta(n+1)$. Thus,

$$\int_0^\infty \frac{x^n}{e^x-1}\,dx = \zeta(n+1)\Gamma(n+1) \tag{A.10}$$

With $n=3$,

$$\zeta(n+1)=\zeta(4)=\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90} \tag{A.11}$$

$$\Gamma(n+1)=n!=3!=6 \tag{A.12}$$

And finally,

$$\int_0^\infty \frac{x^3}{e^x-1}\,dx = \zeta(n+1)\Gamma(n+1)=\frac{\pi^4}{15} \tag{A.13}$$

B Magnetism as a Consequence of Relativity*

As it turns out, the magnetic field we normally think of as a distinct physical phenomena is nothing more than a relativistic view of the electric field of moving charges.[i] In order to see the fundamental symmetry between the electric and magnetic fields, we will conduct a hypothetical experiment using a current-carrying wire and a moving test charge, as shown in Fig B.1. We have a conducting wire with current flowing to the right when viewed from the laboratory reference frame (O). For simplicity, we will assume the current is due to the flow of *positive* charges, spaced evenly with an average separation l^{O} when viewed from the lab frame O.[ii]

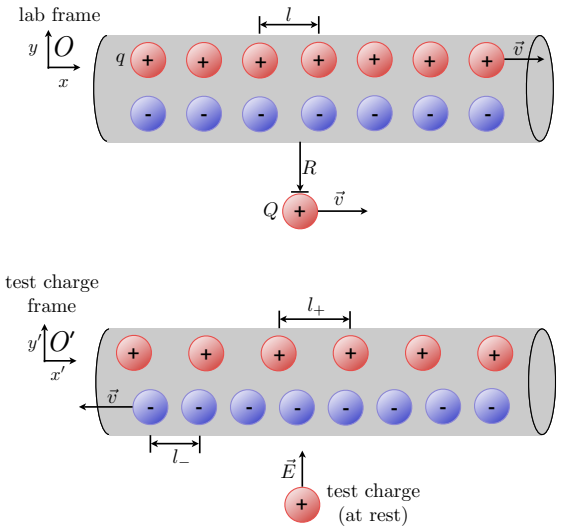

Figure B.1: *An electric current in a wire viewed from the laboratory reference frame (O), and the reference frame of a moving test charge Q (O′). In the test charge frame, the spacing of the positive charges apparently increases while the spacing of the negative charges apparently decreases.*

We know that our conducting wire must be electrically neutral in the laboratory frame, so in addition to the positive charges there must be an equal number of negative ions – the atoms making up the wire – also spaced at a distance l^{O}. Now (still in the laboratory frame) we place a positive test charge Q a distance R from the wire. Since the wire is electrically neutral, there is no force on the test charge. What happens if the test charge is moving? We will give the test charge Q a velocity $\vec{v}$ parallel to the wire, the same velocity with which the positive charges in the wire are moving for simplicity.

What does the now moving test charge experience, viewed from its own reference frame (O′)? Since it is moving in the same direction, with the same velocity, as the positive charges in the wire, *it sees those positive charges as at rest relative to itself*, and *the negative charges as moving to the*

[i] This section follows the treatment of Purcell[**?**] closely.

[ii] Even though we know now that negatively-charged electrons really carry the current, working with positive charges will make the discussion simpler (by avoiding a lot of pesky minus signs), and will not change the analysis in any way.

left with velocity $\vec{v}$.

When the positive charges are viewed from the laboratory frame O, they appear to have an average spacing of l^O, moving at velocity $\vec{v}$. Once we switch to the test charge's frame, the positive charges appear to be at rest – in switching reference frames, the velocity of the positive charges goes from $\vec{v}$ to zero. From special relativity, we know that moving objects undergo a *length contraction*. When we view the spacing l^O of the positive charges in the lab frame O, *we are viewing the contracted length*. In the test charge's frame O', we must un-contract the spacing l^O into the O' frame to figure out what the test charge really sees. If we call the spacing of the positive charges that the moving test charge experiences in its frame O' as $l_+^{O'}$, we can easily relate it to the spacing viewed from the lab frame O:

$$l_+^{O'} = l^O \gamma \tag{B.1}$$

$$l_+^{O'} = \frac{l^O}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{B.2}$$

Since we know $\gamma \geqslant 1$, it is clear that the spacing the test charge sees is *larger* than what we see in the lab frame. Meanwhile, what about the negative charges, which are stationary in the lab frame? The test charge sees from its frame the negative charges moving to the *left* with velocity $\vec{v}$, so their spacing must be *contracted* to figure out the spacing of the negative charges $l_-^{O'}$ the test charge sees:

$$\gamma l_-^{O'} = l^O \tag{B.3}$$

$$l_-^{O'} = \frac{l^O}{\gamma} \tag{B.4}$$

$$l_-^{O'} = l^O \sqrt{1 - \frac{v^2}{c^2}} \tag{B.5}$$

Again, since $\gamma \geqslant 1$, the positive test charge sees a *reduced* spacing of the negative charges. Since the positive and negative charges now no longer appear to have the same spacing when viewed from the test charge's frame, *the test charge sees a net negative charge density*, since there are effectively more negative charges per unit length than positive charges. The presence of a net negative charge density from the test charge's point of view means that it experiences a net attractive force from the wire. From the lab frame, we would not expect any force between the test charge and the wire, but sure enough, a proper relativistic treatment leads us to deduce that a force must in fact be present.

How big is the force? First, we need to figure out the charge density in the wire that the test charge sees. Since we don't want to restrict ourselves to any particular length of wire, we will calculate the number of charges per unit length as viewed in the test charge's frame, $\lambda^{O'}$. How do we

find this? We know that all charges in the wire have charge q, and we know their average spacing. Dividing q by the average spacing for each kind of charge will give us the number of charges per unit length for both positive and negative charges, and subtracting those two will give use the net charge density:

$$\lambda^{O'} = \lambda_+^{O'} - \lambda_-^{O'} \tag{B.6}$$

$$= \frac{q}{l_+^{O'}} - \frac{q}{l_-^{O'}} \tag{B.7}$$

$$= \frac{q}{l^O}\sqrt{1-\frac{v^2}{c^2}} - \frac{q}{l^O}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \tag{B.8}$$

$$= \frac{q}{l^O}\left(\sqrt{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{B.9}$$

This is a bit messy. However, we know that the drift velocity of charges in a conductor is *very* small compared to c ($v_d \sim 10^{-3}\,\mathrm{m/s}$. When $v \ll c$, we can use the following approximations:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \qquad v \ll c \tag{B.10}$$

$$\frac{1}{\gamma} = \sqrt{1-\frac{v^2}{c^2}} \approx 1 - \frac{1}{2}\frac{v^2}{c^2} \qquad v \ll c \tag{B.11}$$

Using these approximations in Eq. B.9, we can come up with a simple expression for $\lambda^{O'}$:

$$\lambda^{O'} = \frac{q}{l^O}\left(\sqrt{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{B.12}$$

$$= \frac{q}{l^O}\left(1 - \frac{1}{2}\frac{v^2}{c^2} - \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)\right) \tag{B.13}$$

$$= -\frac{q}{l^O}\frac{v^2}{c^2} \tag{B.14}$$

Now that we have the charge density of the wire as viewed from the test charge's frame, what is the electrostatic force? The problem is now to find the electric field at a distance R from a long, uniformly charged wire of charge density $\lambda^{O'}$, which is easily found from Gauss' law to be $E = 2k_e\lambda/R$. Substituting, we can immediately write down the electrostatic force experienced by the test charge in its reference frame:

$$|\vec{F}| = Q|\vec{E}| = Q\frac{2k_e|\lambda^{O'}|}{R} = \frac{2k_eQqv^2}{Rlc^2} \tag{B.15}$$

We can simplify this a bit. The current in the wire is the charge q divided by the time it takes the charges to move a unit length, which is $\Delta t = l/v$.[iii] Thus the current can be written as qv/l:

$$|\vec{F}| = Qv\left(\frac{2k_e I}{c^2 R}\right) \tag{B.16}$$

If we make the identification

$$|\vec{B}| = \frac{2k_e I}{c^2 R} = \frac{\mu_o I}{2\pi R} \qquad (c^2 = 1/\mu_o \epsilon_o) \tag{B.17}$$

then we have recovered the magnetic force law:

$$|\vec{F}| = Qv|\vec{B}| \qquad \text{with} \qquad |\vec{B}| = \frac{2k_e I}{c^2 R} \tag{B.18}$$

This is it. A test charge moving near a current-carrying wire experiences a net force proportional to its charge, velocity, and the current in the wire. We have managed to derive the existence of the magnetic field and magnetic force from nothing more than Coulomb's law and special relativity. In the laboratory frame, we typically consider a magnetic field created by the current in the wire, which acts on the test charge to produce a force qvB. What we have shown now is that we find exactly the same force on the test charge by considering it in its own reference frame, thus establishing that **a magnetic field is nothing more than the field of moving charges.**

In some sense, it is remarkable that we can measure magnetic forces due to currents at all. The drift velocity is *miniscule* compared to c, $\frac{v}{c} \sim 10^{-12}$ or so, and γ is barely different from 1, about $1.0{+}10^{-24}$. The magnetic force results from a tiny relativistic correction, certainly, but it is indeed a significant effect in the end because there are truly astronomical numbers of charges per unit length inside conductors. Even though the force per charge is miniscule, they make up for it in numbers. Before moving on, we note that if you repeat this analysis for the more complicated case that the test charge's velocity is *not* the same as the charges in the wire, and *not* parallel, you still arrive at the same result. It just takes quite a bit longer ...

[iii] This just comes from kinematics, we know that the charge covers a distance l according to $l = v\Delta t$.

C

General field transformation rules*

For the curious, we list here the general transformation between E and B fields in different (non-accelerating) frames.

Assume we have two reference frames O and O' whose coordinate axes are all parallel (i.e., x' parallel to x, y' parallel to y, etc.), with frame O' traveling at relative velocity v with respect to frame O along the x' axis. If we have fields E and B in frame O, the fields seen by an observer in frame O' are

$$E'_x = E_x \tag{C.1}$$
$$E'_y = \gamma\left(E_y - vB_z\right) \tag{C.2}$$
$$E'_z = \gamma\left(E_z + vB_y\right) \tag{C.3}$$
$$B'_x = B_x \tag{C.4}$$
$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \tag{C.5}$$
$$B'_z = \gamma\left(B_z + \frac{v}{c^2}E_y\right) \tag{C.6}$$

Note that the components of both E and B parallel to the motion remain unchanged. Two special cases are worth noting. If $E=0$ in O (purely magnetic field in one frame), then

$$\vec{E}' = \vec{v} \times \vec{B}' \tag{C.7}$$

If $B=0$ in O (purely electric field in one frame), then

$$\vec{B}' = -\frac{1}{c^2}\left(\vec{v} \times \vec{E}'\right) \tag{C.8}$$

If either E or B is zero in one frame, the fields in the other frame at a particular point are simply related.