PH253 Lecture 10: Photons, but more so

Compton scattering

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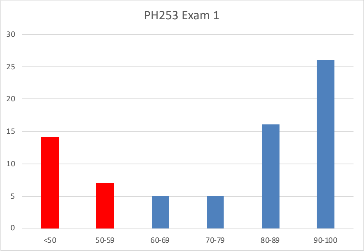
Exam 1

1. Raw average: 71.4%
2. Average excluding results under 60% (21/73): 87.2%
3. Very bimodal distribution ...
4. Exams returned, solution out when makeups done.



Outline

1. Compton Scattering
2. Problems

Last time:

1. Light looks like a particle on large scales ($\gg \lambda$)
2. But it looks like a wave on small scales ($\lesssim \lambda$)
3. Photoelectric effect details supports this idea
4. Photons have momentum by virtue of energy, but no mass
5. Better evidence light = photons? Scattering.

Energy & Momentum for photons:

$$E = hf = \frac{hc}{\lambda} \qquad p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda}$$

Outline

Compton scattering

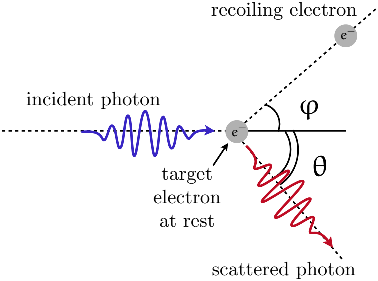
1. if light = particles = photons ...
2. ...scatter the photons off of another particle, e.g., e^-
3. if photon=particle, specific angular dispersion, energy loss
4. Energy loss of photon = red shift = observable
5. classical (EM waves) - incident / scattered photons $\sim$ same f

Compton scattering

An incident photon of frequency f_i, energy $E_i = hf_i$, and momentum $p_i = h/\lambda_i$ strikes an electron (mass m) at rest.

The photon is scattered through an angle θ, and the scattered photon has frequency f_f, energy $E_f = hf_f$, and momentum $p_f = h/\lambda_f$.

Electron recoils at angle φ relative to the incident photon, acquires kinetic energy KE_e and momentum p_e.



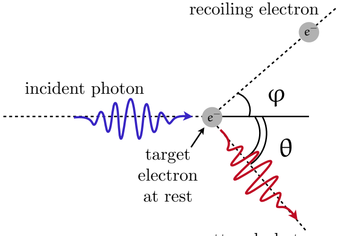
Compton scattering

Conserve energy and momentum:

$$hf_i = hf_f + KE_e = hf_f + \sqrt{m^2c^4 + p_e^2c^2} - mc^2$$

Noted KE_{e^-} is total energy minus its rest energy mc^2.
Conservation of p in both directions:

$$p_i = p_e \cos\varphi + p_f \cos\theta \qquad p_e \sin\varphi = p_f \sin\theta$$



Compton scattering

Solution made simpler by defining dimensionless energy parameters.

Recognizes e^- rest energy mc^2 is the key energy scale

$$\alpha_i = \frac{\text{incident photon energy}}{\text{electron rest energy}} = \frac{hf_i}{mc^2}$$

$$\alpha_f = \frac{\text{scattered photon energy}}{\text{electron rest energy}} = \frac{hf_f}{mc^2}$$

$$\epsilon = \frac{\text{electron kinetic energy}}{\text{electron rest energy}} = \frac{E_e}{mc^2}$$

Compton scattering

Substitutions change our energy and momentum equations to:

$$\alpha_i = \alpha_f + \sqrt{\frac{p_e^2}{m^2c^2} + 1} - 1$$

$$\alpha_i = \alpha_f \cos\theta + \left(\frac{p_e}{mc}\right) \cos\varphi$$

$$\alpha_f \sin\theta = \left(\frac{p_e}{mc}\right) \sin\varphi$$

Compton scattering

Experiment gets incident and scattered photons' energy and angle

So eliminate the electron's momentum p_e and scattering angle φ

Rearrange energy equation, square it, solve for p_e:

$$\alpha_i - \alpha_f + 1 = \sqrt{\frac{p_e^2}{m^2c^2} + 1}$$

$$\frac{p_e^2}{m^2c^2} = \left(\alpha_i - \alpha_f + 1\right)^2 - 1$$

$$p_e^2 = m^2c^2\left(\alpha_i^2 - 2\alpha_i\alpha_f + \alpha_f^2 + 2\alpha_i - 2\alpha_f\right)$$

Compton scattering

$$p_e^2 = m^2c^2\left(\left(\alpha_i - \alpha_f\right)^2 + 2\left(\alpha_i - \alpha_f\right)\right)$$

Now square and add the two momentum equations to eliminate φ

$$\left(\frac{p_e}{mc}\right)\cos\varphi = \alpha_i - \alpha_f\cos\theta \qquad \Longrightarrow \qquad p_e^2\cos^2\varphi = m^2c^2\left(\alpha_i - \alpha_f\cos\theta\right)^2$$

$$\left(\frac{p_e}{mc}\right)\sin\varphi = \alpha_f\sin\theta \qquad \Longrightarrow \qquad p_e^2\sin^2\varphi = m^2c^2\alpha_f^2\sin^2\theta$$

$$p_e^2 = m^2c^2\left(\alpha_f^2\sin^2\theta + \left(\alpha_i - \alpha_f\cos\theta\right)^2\right)$$

$$p_e^2 = m^2c^2\left(\alpha_f^2\sin^2\theta + \alpha_i^2 - 2\alpha_i\alpha_f\cos\theta + \alpha_f^2\cos^2\theta\right)$$

Compton scattering

$$p_e^2 = m^2c^2\left(\alpha_f^2 + \alpha_i^2 - 2\alpha_i\alpha_f\cos\theta\right)$$

Comparing this with our previous equation for p_e^2:

$$p_e^2 = m^2c^2\left(\alpha_i^2 - 2\alpha_i\alpha_f + \alpha_f^2 + 2\alpha_i - 2\alpha_f\right)$$

$$\alpha_i^2 - 2\alpha_i\alpha_f + \alpha_f^2 + 2\alpha_i - 2\alpha_f = \alpha_f^2 + \alpha_i^2 - 2\alpha_i\alpha_f\cos\theta$$

$$-\alpha_i\alpha_f + \alpha_i - \alpha_f = -\alpha_i\alpha_f\cos\theta$$

$$\alpha_i - \alpha_f = \alpha_i\alpha_f\left(1-\cos\theta\right) \qquad \text{or} \qquad \frac{1}{\alpha_f} - \frac{1}{\alpha_i} = 1 - \cos\theta$$

Compton scattering

End result: substitute back for $\alpha = hf/mc^2 = h/\lambda mc$.

$$\frac{1}{\alpha_f} - \frac{1}{\alpha_i} = 1 - \cos\theta$$

$$\frac{\lambda_f mc}{h} - \frac{\lambda_i mc}{h} = 1 - \cos\theta$$

$$\lambda_f - \lambda_i = \Delta\lambda = \frac{h}{mc}\left(1 - \cos\theta\right)$$

This is the *Compton equation.*

h/mc has units of length - the *Compton wavelength*

$\lambda_c = h/mc \approx 2.42 \times 10^{-12}\,\text{m}$.

Scale at which quantum effects dominate

Compton scattering

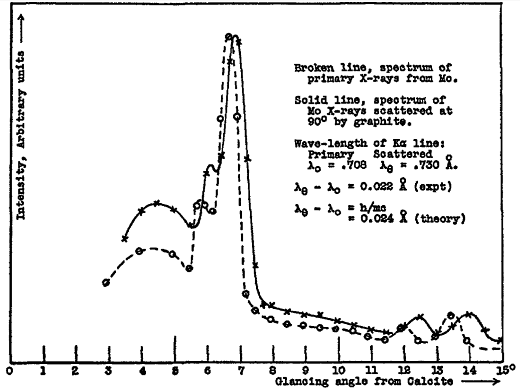


Fig. 4. Spectrum of molybdenum X-rays scattered by graphite, compared with the spectrum of the primary X-rays, showing an increase in wave-length on scattering.

Figure: Original data. Physical Review **21**, 483-502 (1923)

Compton scattering

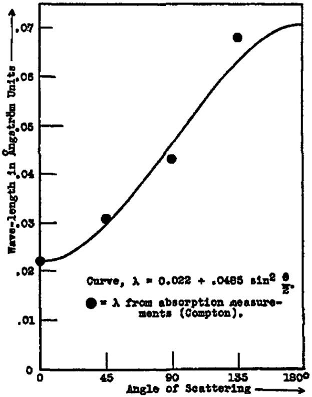


Figure: It works! Physical Review **21**, 483-502 (1923)

Electron kinetic energy

e^- kinetic energy = difference between incident & scattered photons:

$$KE_e = hf_i - hf_f = \alpha_i mc^2 - \alpha_f mc^2 = \left(\alpha_i - \alpha_f\right) mc^2$$

Solving the Compton equation for α_f, we have

$$\alpha_f = \frac{\alpha_i}{1 + \alpha_i \left(1 - \cos\theta\right)}$$

Combining these two equations

$$KE_e = \left(\alpha_i - \alpha_f\right) mc^2 = mc^2 \left(\alpha_i - \frac{\alpha_i}{1 + \alpha_i \left(1 - \cos\theta\right)}\right)$$

$$\epsilon = \frac{KE_e}{mc^2} = \frac{\alpha_i^2 \left(1 - \cos\theta\right)}{1 + \alpha_i \left(1 - \cos\theta\right)}$$

Compton scattering

$$KE_e = (\alpha_i - \alpha_f)\, mc^2 = mc^2 \left(\alpha_i - \frac{\alpha_i}{1 + \alpha_i (1 - \cos\theta)} \right)$$

$$\epsilon = \frac{KE_e}{mc^2} = \frac{\alpha_i^2 (1 - \cos\theta)}{1 + \alpha_i (1 - \cos\theta)}$$

e^- kinetic energy can only be a fraction of the incident photon's energy

There will *always* be some energy left over for a scattered photon.

Means that *a stationary, free electron cannot absorb a photon!*

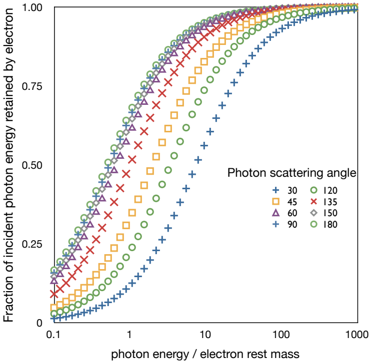
Absorption can only occur if the electron is bound to, e.g., a nucleus which can take away a bit of the net momentum and energy.

Compton scattering

1. Shift in wavelength $\Delta\lambda$ independent of photon energy
2. Shift in photon *energy* is not
3. Change in photon energy is equal to e^- KE
4. Strongly dependent on the incident photon energy!
5. Relevant energy scale set by the ratio of the incident photon energy to e^- rest energy
6. If ratio is large, the fractional shift in energy is large
7. When the incident photon energy $\sim e^-$ rest energy, Compton scattering significant
8. $mc^2 \approx 511\,\text{keV}$, hard x-rays or gamma rays

Compton scattering

Fraction of the photon energy retained by the e^- vs. incident photon energy for various photon scattering angles.



Compton scattering

Maximum electron energy or photon energy shift? Set $d\epsilon/d\theta = 0$:

$$\epsilon = \frac{KE_e}{mc^2} = \frac{\alpha_i^2 \left(1 - \cos\theta\right)}{1 + \alpha_i \left(1 - \cos\theta\right)}$$

$$\frac{d\epsilon}{d\theta} = \alpha_i^2 \left[\frac{-\alpha_i \sin\theta \left(1 - \cos\theta\right)}{\left(1 + \alpha_i \left(1 - \cos\theta\right)\right)^2} + \frac{\sin\theta}{1 + \alpha_i \left(1 - \cos\theta\right)}\right] = 0$$

$$0 = \sin\theta \left[-\alpha_i + \alpha_i \cos\theta + 1 + \alpha_i \left(1 - \cos\theta\right)\right]$$

$$0 = \sin\theta \qquad \Longrightarrow \qquad \theta = \{0, \pi\}$$

$\theta = 0$ can be discarded, this corresponds to the photon going right through the electron

At $\theta = \pi$, backward scattering of photon, electron has max energy

Compton scattering

Maximum energy of the electron at $\theta = \pi$ (photon backscattered)

$$KE_{\text{max}} = hf_i \left(\frac{2\alpha_i}{1 + 2\alpha_i} \right)$$
$$\epsilon = \alpha_i \left(\frac{2\alpha_i}{1 + 2\alpha_i} \right) = \frac{2\alpha_i^2}{1 + 2\alpha_i}$$

Max e^- KE at most a fraction of the incident photon energy - absorption cannot occur for free electrons

Compton scattering

Compton wavelength sets fundamental limitation on measuring the position of a particle.

- Depends on the mass m of the particle.
- Can measure the position of a particle by bouncing light off it
- But! Need short wavelength for accuracy.
- That means higher p and E for the photon!
- (Which disturbs position . . . uncertainty)
- If photon energy $> mc^2$, when it hits particle being measured there is enough energy to create a new particle of the same type!
- Meaning you still don't know where the original one is.
- Or if photon energy $> 2mc^2$, photon can decay into particle-antiparticle pair

Compton scattering

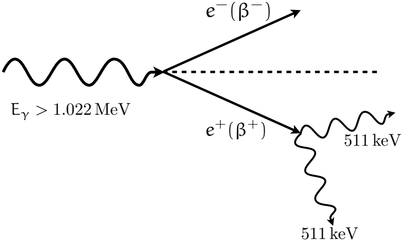


Figure: Pair production

A photon can "decay" into an electron and a positron (electron antiparticle). Try to measure electron with high energy photon? Now you have 3 particles.

Outline

Problem

Show that it is impossible for a photon striking a free electron to be absorbed and not scattered.

1. Choose frame where e^- is at rest (free choice)
2. Conserve E & p, show something impossible is implied
3. Before: photon of energy hf and momentum h/λ
4. Before: electron with rest energy mc^2
5. After: e^-, $E_{\text{tot}} = (\gamma - 1) + mc^2 = \sqrt{p^2c^2 + m^2c^4}$ & $p_e = \gamma m v$
6. p conservation: absorbed photon's momentum transferred to e^-
7. e^- must continue along the same line that the incident photon traveled
8. $\Longrightarrow$ 1D problem

Problem

Enforcing conservation of energy and momentum:

$$hf + mc^2 = \sqrt{p^2c^2 + m^2c^4} \qquad \text{energy conservation variant 1}$$

$$hf + mc^2 = \gamma mc^2 \qquad \text{energy conservation variant 2}$$

$$\frac{h}{\lambda} = p_e = \gamma mv \qquad \text{momentum conservation}$$

Use conservation of p to put e^- momentum in terms of the photon frequency:

$$\frac{h}{\lambda} = p_e \qquad \Longrightarrow \qquad \frac{hc}{\lambda} = hf = p_ec$$

Problem

Enforcing conservation of energy and momentum:

Use this in energy conservation equation to eliminate p_e, square both sides, and collect terms.

$$(hf + mc^2)^2 = \left(\sqrt{p^2c^2 + m^2c^4}\right)^2 = \left(\sqrt{h^2f^2 + m^2c^4}\right)^2$$

$$h^2f^2 + 2hfmc^2 + m^2c^4 = h^2f^2 + m^2c^4$$

$$2hfmc^2 = 0 \qquad \Longrightarrow \qquad f = 0 \qquad \Longrightarrow \qquad p_e = v = 0$$

The only way a photon can be absorbed by the stationary electron is if its frequency is zero, *i.e., if there is no photon to begin with!*

Problem

Use 2nd consv. E equation + consv. p to get same conclusion:

$$hf = \frac{hc}{\lambda} = \gamma mc^2 \qquad \text{energy conv. variant 2}$$

$$\frac{h}{\lambda} = \gamma mv \qquad \text{momentum consv.}$$

$$\Longrightarrow \qquad \gamma mvc = \frac{hc}{\lambda} = \gamma mc^2 \qquad \Longrightarrow \qquad \gamma v = \gamma c$$

Implies $v = c$ for e^-, which is not possible (requires infinite energy)
Or, requires $\gamma = 0$, which is also not possible

Problem 2

A proton is uniformly accelerated in a van de Graaff accelerator through a potential difference of 700 kV. The length of the linear accelerating region is 3 m. **(a)** Compute the ratio of the radiated energy to the final kinetic energy. **(b)** Show that for a particle moving in a linear accelerator the rate of radiation of energy is

$$\frac{dU}{dt} = \frac{q^2}{6\pi\epsilon_o m^2 c^3}\left(\frac{dK}{dx}\right)^2 \tag{1}$$

where K is the kinetic energy.

Problem 2

The distance covered must be $s = \frac{1}{2}at^2$ and the velocity $v = at$, which we can combine to give $s = \frac{1}{2}vt$.

That implies $t = 2s/v$ and $a = v/t = v^2/2s$

Plug in to energy equation:

$$U_{\text{rad}} = P_{\text{rad}}\Delta t = \frac{q^2 a^2}{6\pi\epsilon_o c^3}\Delta t = \frac{q^2 v^3}{12\pi\epsilon_o c^3 s}$$

Problem 2

Now use conservation of energy: potential energy change due to ΔV same as kinetic energy change.

$$K = \frac{1}{2}mv^2 = q\Delta V \qquad \Longrightarrow \qquad v = \sqrt{\frac{2q\Delta V}{m}}$$

Put it together:

$$\frac{U_{\text{rad}}}{K} = \frac{1}{\frac{1}{2}mv^2}\frac{q^3v^3}{12\pi\epsilon_o c^3 s} = \frac{q^3 v}{6\pi\epsilon_o mc^3 s} = \frac{q^3}{6\pi\epsilon_o mc^3 s}\sqrt{\frac{2q\Delta V}{m}}$$

$$\frac{U_{\text{rad}}}{K} = \frac{q^3}{6\pi\epsilon_o c^3 s}\sqrt{\frac{2q\Delta V}{m^3}} \approx 1.31 \times 10^{-20}$$

Problem 2

Radiation losses in linear accelerators are utterly negligible.

The rate of energy loss can be related to the kinetic energy gained per unit distance (dK/dx):

$$K = \frac{1}{2}mv^2$$
$$\frac{dK}{dx} = mv\frac{dv}{dx} = mv\frac{dv}{dt}\frac{dt}{dx} = ma$$

Last step, chain rule and $dt/dx = 1/v$. With $a = \frac{1}{m}\frac{dK}{dx}$, plug into power equation

$$\frac{dU}{dt} = P_{\text{rad}} = \frac{q^2a^2}{6\pi\epsilon_o m^2c^3} = \frac{q^2}{6\pi\epsilon_o m^2c^3}\left(\frac{dK}{dx}\right)^2$$