PH253 Lecture 15: Schrödinger's equation. Still.

1-D potentials

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Image of the day

High-resolution electron microscope image of a $Co_{1.5}Ti_{0.5}FeGe$ alloy sample. The bright dots are individual atoms. The inset is a Fourier transform of the image, indicating the hexagonal symmetry.

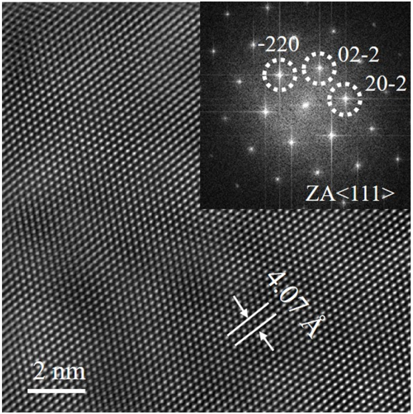
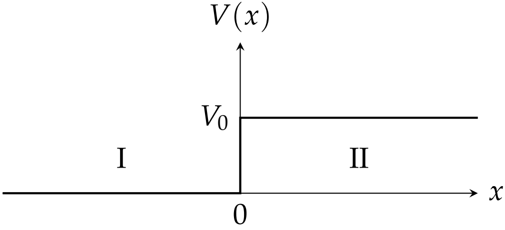


Figure: UA Physics / LeClair group / Phys. Rev. Materials 3, 114406 (2019)

Outline

1. Potential step
2. Particle in a box
3. Observables and operators

Potential step from last time



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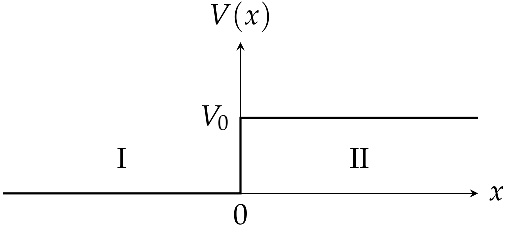

$$\psi(x)_I = e^{ikx} + Re^{-ikx} \qquad \psi(x)_{II} = Te^{iqx}$$

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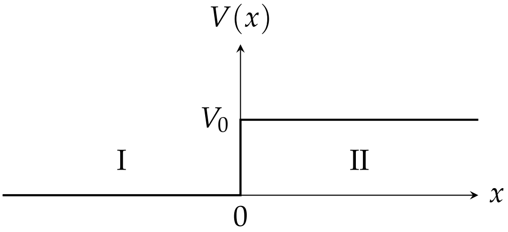

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6. $E = V_0$? $P_{\text{trans}} = 0$, perfect reflection

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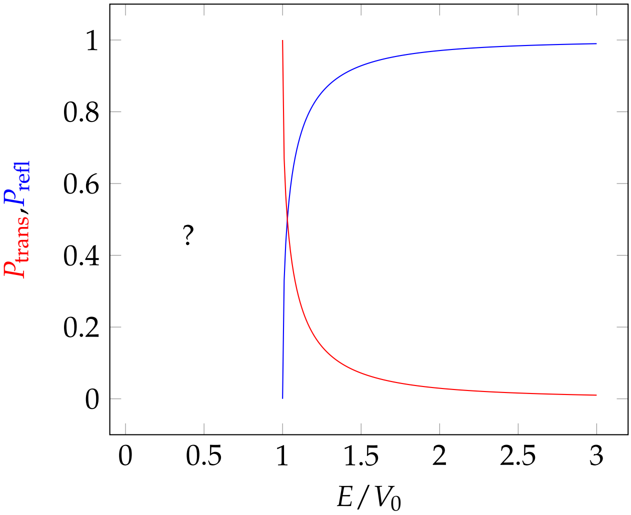

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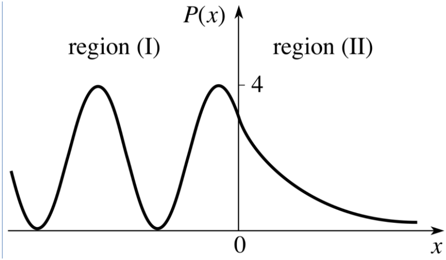

Figure: http://www.met.reading.ac.uk/pplato2/h-flap/phys11_1.html

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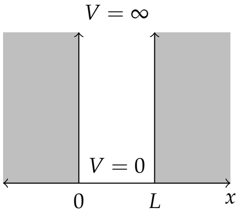

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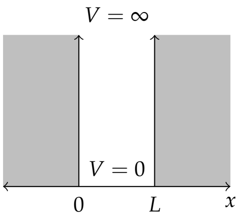
2. Like book derivation of blackbody radiation



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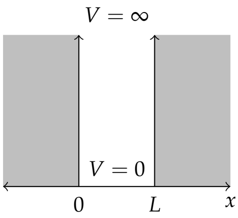

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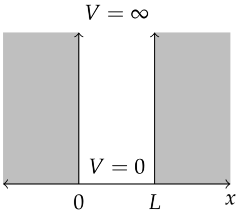

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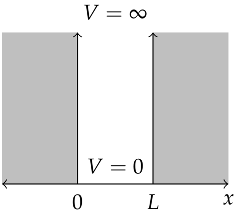
5. Implies boundary condition $\psi(0) = \psi(L) = 0$!



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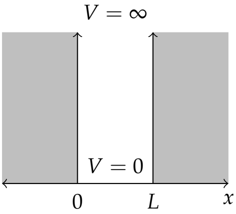

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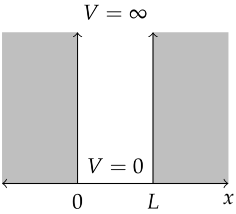

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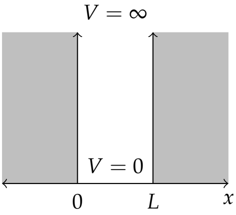

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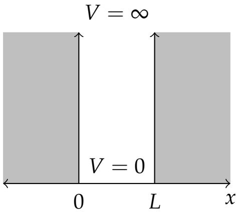

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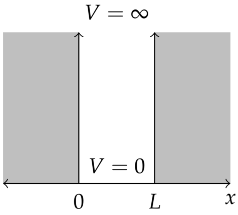

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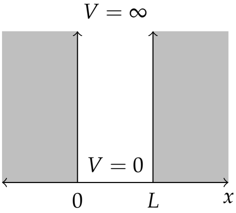
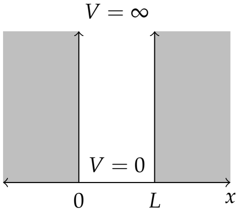
Now set up Schrödinger's equation in box + boundary conditions.



Boundary conditions are key

Schrödinger equation with $V = 0$:

$$\frac{d^2\psi}{dx^2} = -\left(\frac{2mE}{\hbar^2}\right)\psi = -k^2\psi \qquad \left(\text{with } k^2 = \frac{2mE}{\hbar^2}\right)$$



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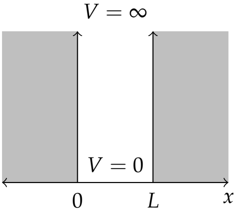

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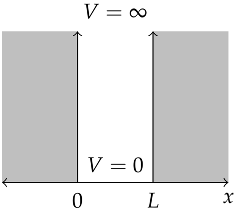

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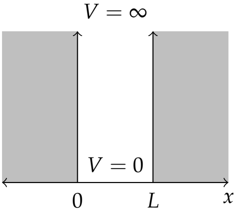
How to find constants? Boundary conditions! $\psi(0) = \psi(L) = 0$



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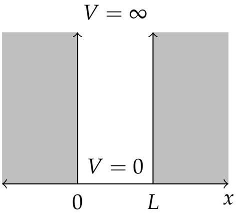

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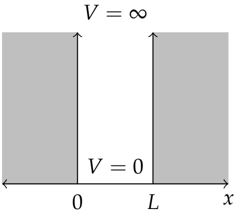

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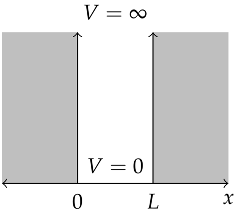

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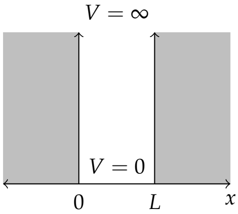
Recall: bound states have ψ purely real



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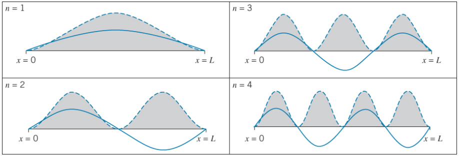

Figure: Krane, *Modern Physics*. Solid: ψ. Dashed: $|\psi|^2$.

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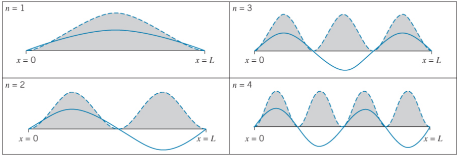

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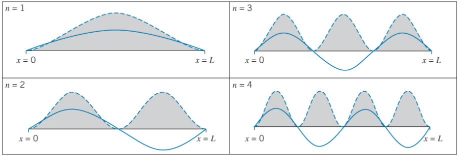

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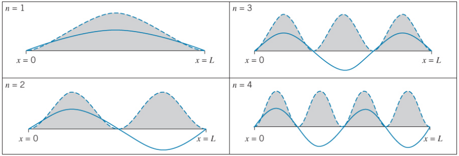
$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2mL^2} = \frac{n^2 h^2}{8mL^2}$$



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Discrete energies

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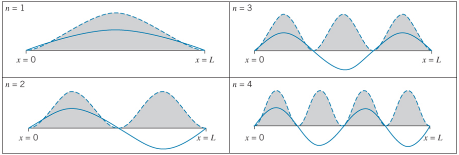

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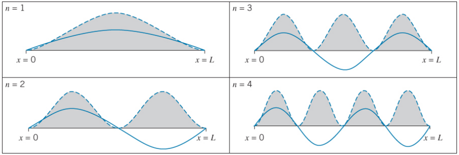

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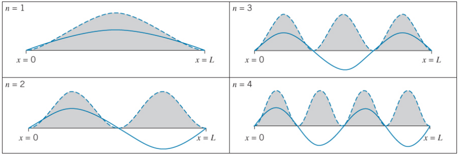

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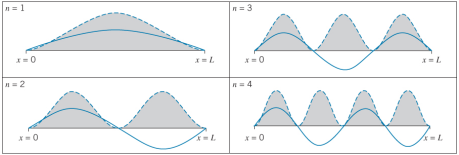
3. Where we went wrong with free particle: no boundary conditions



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Discrete energies

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1. Since V is independent of time, time variation is $e^{-iE_n t/\hbar}$
2. Big picture: *confinement* leads to quantized energy levels.
3. Where we went wrong with free particle: no boundary conditions
4. What are the C_n? We have not normalized the wave function ...

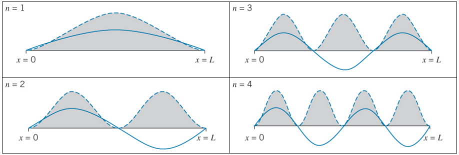

Figure: Krane, *Modern Physics*. Solid: ψ. Dashed: $|\psi|^2$.

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$$1 = \int_0^L |\psi(x)|^2 \, dx = \int_0^L C_n^2 \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= C_n^2 \left[\frac{x}{2} - \frac{4L}{n\pi} \sin\left(2\frac{n\pi x}{L}\right)\right] \Bigg|_0^L = C_n^2 \frac{L}{2}$$

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8. What about position, and how to use these new operators?

Outline

1. Potential step
2. Particle in a box
3. Observables and operators

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$$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^* x \psi \, dx = \int_{-\infty}^{+\infty} x |\psi(x)|^2 \, dx = \int_{-\infty}^{+\infty} x P(x), dx$$

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But! Can now formalize uncertainty principle & actually calculate.

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3. Measurement? $x_{\text{best}} = \langle x \rangle \pm \langle x^2 \rangle = (0.500 \pm 0.289) L$ – broad

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3. We know $p^2 = 2mE$ for a free particle, must be true in box too ...

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3. Not unreasonable: uncertainty up as n (and E_n) increase

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