PH253 Lecture 17: Hydrogen atom
ground state

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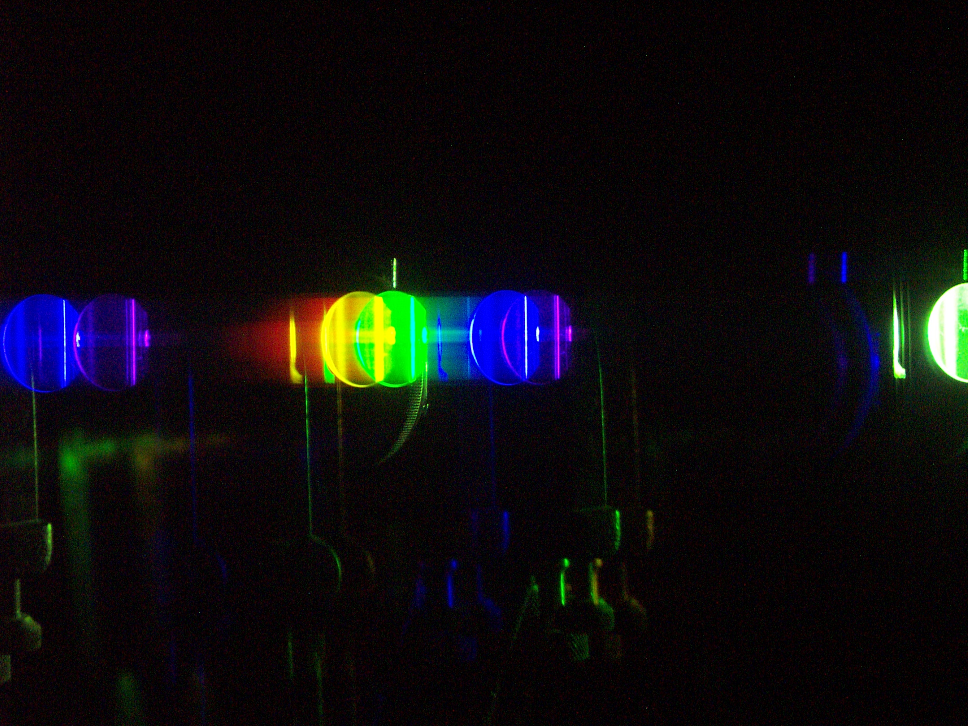
The University of Alabama

Spring 2020



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- HW2: Know how to use the uncertainty principle, particularly for microscope resolution.

Outline

1. Now in 3D

How do we use Schrödinger's equation in 3D?

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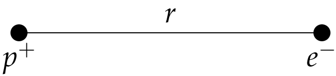

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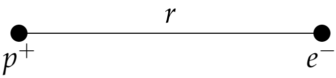

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$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = 0 \qquad \text{-or-} \qquad \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \left(\frac{e^2}{4\pi\epsilon_o r} - E\right)\psi = 0$$

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- Just like we did to separate time-dependent Schrödinger …

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- Try Ae^{-cr} – fits all conditions

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- Can already see exponentials will turn our diff. eq. into algebra …

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- Valid for *all* r? Then r, $1/r$, constant terms equate *separately*

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- Decay constant is Bohr radius a_o! $\quad \psi(r) = e^{-r/a_o}$

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- This is the hydrogen 1s state. What about 2, and p, d, f?

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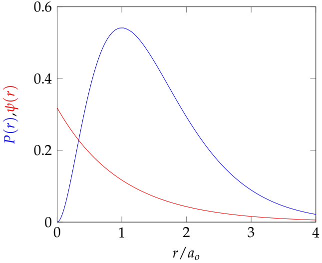

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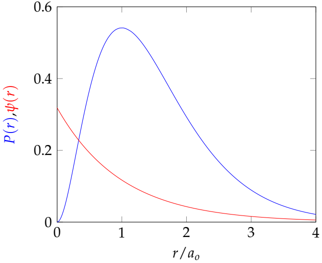

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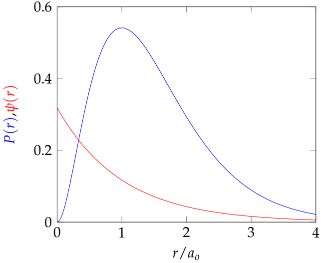

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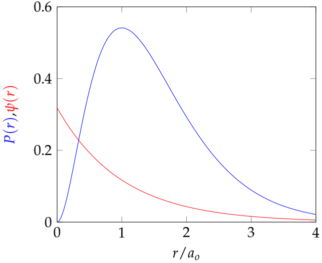

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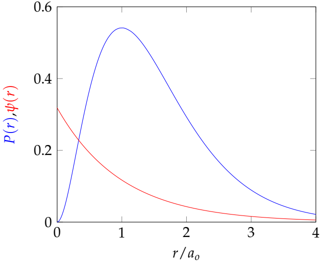

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- Again: in spite of *distribution*, measurement yields 1 e^- at one spot

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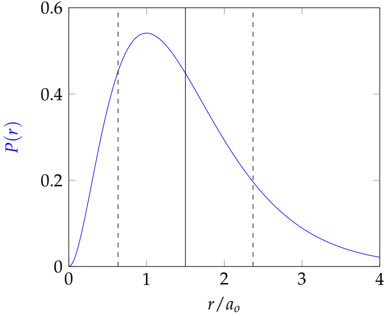

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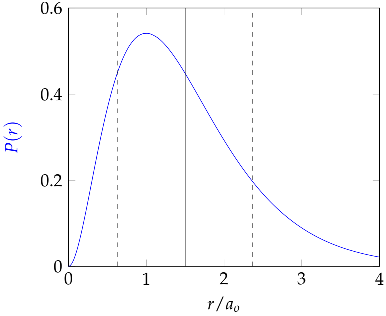

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- Need fewer assumptions, better mechanics, more math