PH253 Lecture 4: relativity

Lorentz Transformations, Dynamics

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Outline

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Lorentz transformations between coordinate systems

$$x' = \gamma\,(x - vt) \quad \text{or} \quad x = \gamma\,(x' + vt') \tag{1}$$

$$y' = y \tag{2}$$

$$z' = z \tag{3}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{or} \quad t = \gamma\left(t' + \frac{vx'}{c^2}\right) \tag{4}$$

Here (x, y, z, t) is the position and time of an event as measured by an observer in O stationary to it. A second observer in O', moving at velocity v along the x axis, measures the same event to be at position and time (x', y', z', t').

Velocity Addition

We have an observer in a frame O, and a second observer in another frame O$'$ who are moving relative to each other at a velocity v. Both observers measure the velocity of another object in their own frames (v_{obj} and v'_{obj}). We can relate the velocities measured in the different frames as follows:

$$v_{\text{obj}} = \frac{v + v'_{\text{obj}}}{1 + \frac{v v'_{\text{obj}}}{c^2}} \qquad v'_{\text{obj}} = \frac{v_{\text{obj}} - v}{1 - \frac{v v_{\text{obj}}}{c^2}} \tag{5}$$

Again, v_{obj} is the object's velocity as measured from the O reference frame, and v_{obj} is its velocity as measured from the O$'$ reference frame.

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Decide if you should add or subtract velocities, then pick the formula.

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$$\frac{dt'}{dt} = \frac{d}{dt}\left(\gamma\left(t - \frac{vx}{c^2}\right)\right) = \gamma\left(1 - \frac{vv_x}{c^2}\right) \tag{8}$$

Velocity Transformations?

Put it together:

$$v'_y = \frac{v_y}{\gamma\left(1 - \frac{vv_x}{c^2}\right)} \tag{9}$$

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Transverse velocities do change! Similar expression along z.

This is a homework problem.

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Expect this for light, but $v_{\text{light}} >> v_{\text{sound}}$ ... harder to observe

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- In stationary frame (e.g., next to light source), period is $T = \frac{\lambda}{c}$
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- *And*, crests of waves are coming faster: $v_w' = v_w + v_o$

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Moving *toward* source at v_o shortens wavelength λ toward blue

Similarly, moving *away* source at v_o lengthens wavelength λ toward red ($v_o < 0$)

Doppler shift?

Red shift: ratio of shifted wavelength to what a stationary observer sees:

$$z = \frac{\lambda'}{\lambda} = \sqrt{\frac{c + v_o}{c - v_o}} \tag{12}$$

- $v_o > 0$ when approaching, $v_o < 0$ when receding.

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Relativity principle: makes no difference if source or observer is moving.

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$$\text{proper velocity} = \eta = \left(\frac{\text{rate of change of posn. in } \textit{observer's} \text{ frame}}{\text{proper time measured by moving observer}}\right)$$

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$$\eta = \frac{dl}{dt_p} = \frac{dl/dt}{dt_p/dt} = \frac{v}{dt_p/dt} \tag{13}$$

$$\frac{dt_p}{dt} = \frac{d}{dt}\left(t/\gamma\right) = \frac{1}{\gamma} \tag{14}$$

$$\Longrightarrow \eta = \frac{v}{\sqrt{1-v^2/c^2}} = \gamma v \tag{15}$$

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What about mass? Modern view: *invariant* - all agree on rest mass

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Energy

The work-energy theorem relates the change in kinetic energy of a particle to the work done on it by an external force: $\Delta K = W = \int F\,dx$. Writing Newton's second law as $F = dp/dt$, show that $W = \int v\,dp$ and integrate by parts to obtain the result

$$K = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2$$

Energy

We are concerned with finding the kinetic energy of a particle accelerated from rest, starting with zero kinetic energy, so we may simply write $K = W = \int F\,dx$ and drop the Δ. Using what we have, and noting $v = dx/dt$:

$$K = \int_{v=0}^{v=v} F\,dx = \int_0^v \frac{dp}{dt}\,dx = \int_0^v dp\,\frac{dx}{dt} = \int_0^v v\,dp \tag{16}$$

Energy

We can integrate by parts. Let $f = dp$ and $g' = dx/dt$, and recall integration by parts gives $\int fg' = fg - \int f'g$.

$$K = \int_0^v v\,dp = pv\Big|_0^v - \int_0^v p\,dv = pv - \int_0^v p\,dv \tag{17}$$

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$$K = \frac{mv^2}{\sqrt{1 - v^2/c^2}} - \int_0^v \frac{mv}{\sqrt{1 - v^2/c^2}}\,dv \tag{18}$$

$$= \frac{mv^2}{\sqrt{1 - v^2/c^2}} + mc^2\sqrt{1 - v^2/c^2}\Big|_0^v \tag{19}$$

Energy

Continuing:

$$K = \frac{mv^2}{\sqrt{1 - v^2/c^2}} + mc^2\sqrt{1 - v^2/c^2} - mc^2 \tag{20}$$

$$= \frac{mv^2 + mc^2\left(1 - v^2/c^2\right)}{\sqrt{1 - v^2/c^2}} - mc^2 \tag{21}$$

$$= \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right) mc^2 \tag{22}$$

$$\Longrightarrow K = (\gamma - 1)\, mc^2 \tag{23}$$

Energy

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Plug this in: $K = (\gamma - 1)\, mc^2 \approx \left(1 + \frac{v^2}{c^2} - 1\right) mc^2 = \frac{1}{2} mv^2$

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- Takes infinite energy to reach light speed, not happening
- Only massless objects (light) can travel at c
- What about second term, independent of speed?!?

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- Or, $E_{tot} = K + mc^2 = \gamma mc^2$ and $E_{rest} = mc^2$
- Bodies have *rest energy* associated with mass – mass-energy equivalent!

Relativistic energy-momentum equations

Can go further ...

$$E_{tot}^2 - (pc)^2 = \left(mc^2\right)^2 \tag{24}$$

$$E_{tot}v = pc^2 \tag{25}$$

Better form: relativistic $E - p$ relationship

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c is really a speed limit for any massive particle