PH253 Lectures 5-8: radiation

Accelerating charges, radiation, and electromagnetic waves

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Outline

Outline

Model and ingredients

How accelerating charges emit radiation?

What is the spectrum of radiation emitted from a hot object, and why do they glow at all?

In short, individual charges in atoms acquire random thermal energy, which causes them to oscillate, which causes them to radiate.

1. Figure out the field from moving charges
2. Find the radiation emitted from accelerating charges
3. From radiation, find radiation reaction force that must be present
4. Compute the equation of motion and energy of oscillating charges
5. Model a hot object as random oscillators excited thermally
6. Realize the result is silly, consider Planck's hypothesis …

Outline

Electric fields in different reference frames

1. Charge is *invariant* - same for all observers.
2. What about the fields?
3. Consider a capacitor:

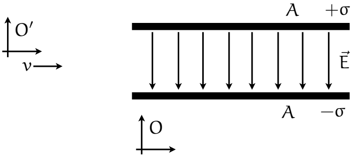


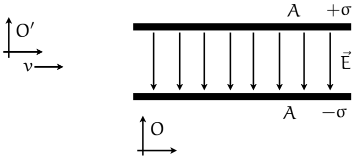
Figure: **(left)**: An observer in O' travels at velocity v perpendicular to the electric field created by a capacitor in frame O. **(right)** An observer in O' travels at velocity v parallel to the electric field created by a capacitor in frame O.

Electric fields in different reference frames

In the capacitor's reference frame O, we know that the field between the plates is

$$E = \frac{\sigma}{\epsilon_o} = \frac{Q}{A\epsilon_o}$$

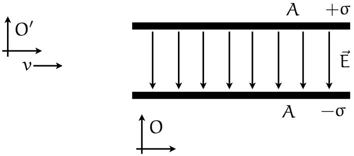
Because total charge on each plate Q is σA.



Electric fields in different reference frames

1. observer travels perpendicular to $\vec{E}$?
2. Dimensions along the direction of motion shortened by a factor γ.
3. Thus *area of plates* in O' smaller by factor γ.
4. Since charge is constant smaller plates means *a larger apparent charge density*!
5. Thus, the observer in O' must see a charge density

$$\sigma' = \gamma\sigma$$

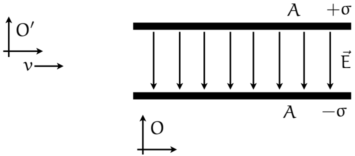


Electric fields in different reference frames

This means the electric field in the observer's frame must be

$$E' = \frac{\sigma'}{\epsilon_o} = \gamma \frac{\sigma}{\epsilon_o} = \gamma E \qquad \left(\vec{v} \perp \vec{E}\right)$$

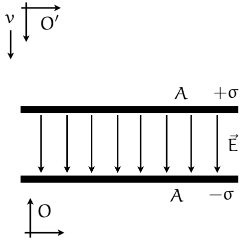
$\vec{E}$ for motion perpendicular to the field is *enhanced* by a factor γ.



Electric fields in different reference frames

1. How about motion parallel to the field?
2. Now the *spacing* of the capacitor is contracted!
3. *Area* of the plates is the same, so σ is too.
4. $\vec{E}$ doesn't depend on the spacing (C does), so field is the same!

$$E' = E \qquad \left(\vec{v} \parallel \vec{E}\right)$$



Electric fields in different reference frames

1. Nothing special about the field of capacitor, just like any other electric field.
2. Derived general transformation of $\vec{E}$ between different reference frames:

$$E'_{\perp} = \gamma E_{\perp}$$
$$E'_{||} = E_{||}$$

1. Components of $\vec{E} \perp \vec{v}$ increased by a factor γ
2. Components of $\vec{E} \parallel \vec{v}$ are unaffected
3. Holds only for charges that are stationary in one of the two frames
4. Moving in both frames? You get $\vec{B}$!
5. Force transforms in the same way since $\vec{F} = q\vec{E}$

Outline

Field from a charge moving at constant velocity

1. Charge q traveling at velocity v along x in O'
2. Charge's rest frame is O.
3. In O the charge is at rest, in O' the charge is in motion at constant v
4. The $\perp$ (z) and $\|$ (x) components of $\vec{E}$ transform differently
5. $\Longrightarrow$ magnitude and orientation of $\vec{E}$ will be different in O'.

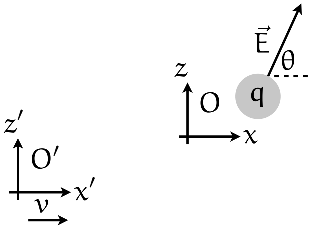


Figure: A charge is at rest in frame O, while frame O' moves with velocity v and angle θ

Field from a charge moving at constant velocity

In frame O, the charge is at rest, so the field at a distance r from the origin measured in O is:[1]

$$E = \frac{kq}{r^2}$$

Broken down by components, we have (noting E_y and E_z are the same)

$$E_x = \frac{kq}{r^2}\cos\theta = \frac{kq}{x^2+z^2}\frac{x}{\sqrt{x^2+z^2}} = \frac{kqx}{\left(x^2+z^2\right)^{3/2}}$$

$$E_z = \frac{kqz}{\left(x^2+z^2\right)^{3/2}}$$

[1] For convenience, we use $k = 1/4\pi\epsilon_o$ for now.

Field from a charge moving at constant velocity

1. In O', the charge is moving at constant velocity v
2. To find the field in O' need to transform coordinates first!

$$x = \gamma\left(x' - vt'\right)$$
$$z = z'$$
$$t = \gamma\left(t' - \frac{vx'}{c^2}\right)$$
$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

1. Now we know field transforms too!
2. E_x is constant, E_z larger by a factor γ:

$$E'_x = E_x$$
$$E'_z = \gamma E_z$$

Field from a charge moving at constant velocity

Using the field transformation and the Lorentz transformations, we can find field in O' for each component:

$$E'_x = E_x = \frac{kqx}{\left(x^2+z^2\right)^{3/2}} = \frac{kq\gamma\left(x'-vt'\right)}{\left(\gamma^2\left(x'-vt'\right)^2+z'^2\right)^{3/2}}$$

$$E'_z = \gamma E_z = \frac{kq\gamma z}{\left(x^2+z^2\right)^{3/2}} = \frac{kq\gamma z'}{\left(\gamma^2\left(x'-vt'\right)^2+z'^2\right)^{3/2}}$$

What a mess ...

Field from a charge moving at constant velocity

Main interest here is to find the difference between the electric field observed by the moving and stationary observer at the same location (i.e., when their origins overlap).

We aren't worried about time dependence, simultaneity, or propagation delays.

Thus, consider $t = t' = 0$ only, which simplifies things

$$E'_x = \frac{kq\gamma x'}{\left(\gamma^2 x'^2 + z'^2\right)^{3/2}}$$

$$E'_z = \frac{kq\gamma z'}{\left(\gamma^2 x'^2 + z'^2\right)^{3/2}}$$

Field from a charge moving at constant velocity

We can already notice that the angle of the field in frame O' is

$$\tan\theta' = \frac{E'_z}{E'_x} = \frac{z'}{x'} \tag{1}$$

1. The field in O' points along the radial direction
2. I.e., E' makes the same angle with the x' axis that r' does.
3. E' points radially outward from the *instantaneous position* of q.

Given both components of the field in E', finding the magnitude of the field is just algebra ...

Field from a charge moving at constant velocity

Algebra ensues.

$$E'^2 = E'^2_x + E'^2_z = \frac{k^2q^2\gamma^2x'^2}{\left(\gamma^2x'^2+z'^2\right)^3} + \frac{k^2q^2\gamma^2z'^2}{\left(\gamma^2x'^2+z'^2\right)^3} = k^2q^2\gamma^2\left[\frac{x'^2+z'^2}{\left(\gamma^2x'^2+z'^2\right)^3}\right]$$

$$= k^2q^2\gamma^2r'^2\left[\frac{1}{\left(\gamma^2x'^2+z'^2\right)^3}\right] \qquad \left(r'^2 = x'^2+z'^2\right)$$

$$= \frac{k^2q^2\gamma^2r'^2}{\gamma^6}\left[\frac{1}{\left(x'^2+z'^2/\gamma^2\right)^3}\right] \qquad \text{factor } \gamma^2 \text{ from denominator}$$

$$= \frac{k^2q^2r'^2}{\gamma^4}\left[\frac{1}{\left(x'^2+z'^2-\left(v^2/c^2\right)z'^2\right)^3}\right] \qquad \left(\frac{1}{\gamma^2} = 1-\frac{v^2}{c^2}\right)$$

$$= \frac{k^2q^2r'^2}{\gamma^4}\frac{1}{\left(x'^2+z'^2\right)^3}\frac{1}{\left[1-\frac{v^2}{c^2}\frac{z'^2}{x'^2+z'^2}\right]^3}$$

Field from a charge moving at constant velocity

1. Still a mess, but note $z'/\sqrt{x'^2+z'^2} = \sin\theta'$ and $r'^2 = x'^2 + z'^2$

$$E'^2 = \frac{k^2 q^2 r'^2}{\gamma^4} \frac{1}{\left(x'^2+z'^2\right)^3} \frac{1}{\left[1 - \frac{v^2}{c^2}\frac{z'^2}{x'^2+z'^2}\right]^3}$$

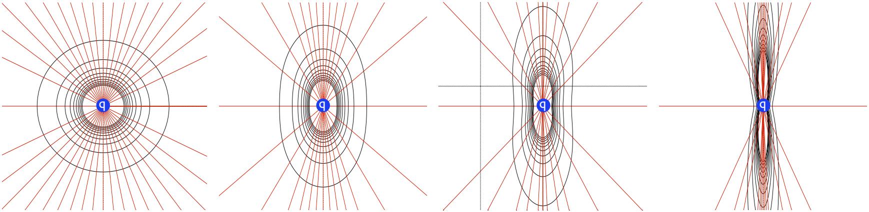
$$E'^2 = \frac{k^2 q^2}{\gamma^4 r'^4} \frac{1}{\left[1 - \frac{v^2}{c^2}\sin^2\theta'\right]^3}$$

$$E'^2 = \frac{k^2 q^2}{r'^4} \frac{\left(1 - \frac{v^2}{c^2}\right)^2}{\left[1 - \frac{v^2}{c^2}\sin^2\theta'\right]^3} \qquad \text{(substitute definition of } \gamma\text{)}$$

$$\Longrightarrow \qquad E' = \frac{kq}{r'^2} \frac{1 - \frac{v^2}{c^2}}{\left[1 - v^2\sin^2\theta'/c^2\right]^{3/2}}$$

Field from a charge moving at constant velocity

1. Finally, field of a moving charge!
2. Field lines end up being "squashed" along the direction of motion
3. Field higher along $\perp$ (z') direction, now *axial*
4. "Relativistic compression" of field lines
5. At all v inverse square law, isotropic only at very low speeds.



$v = 0$ $\quad$ $v = 0.75c$ $\quad$ $v = 0.9c$ $\quad$ $v = 0.99c$

Figure: $\vec{E}$ (red) and contours of constant $\vec{E}$ (black) for a point charge moving at various velocities.

Field from a charge moving at constant velocity

1. As v approaches c, the field is more and more directional
2. Along the horizontal axis ($z'=0$, $\theta=0$), $\vec{E}$ is reduced by a factor γ^2 compared to a stationary charge

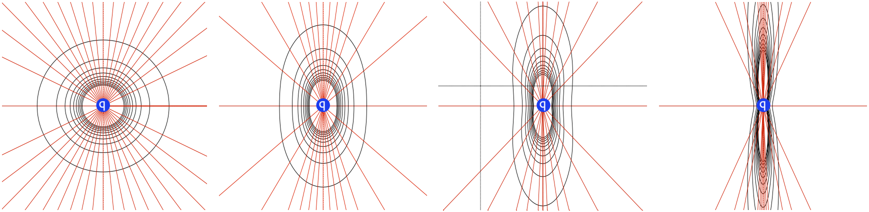
$$E'_{x'} = \frac{kq}{\gamma^2 r'^2} \qquad \text{(along } x'\text{)}$$

Along the vertical axis ($x'=0$, $\theta=90°$), $\vec{E}$ is *enhanced* by a factor γ

$$E'_{y'} = \frac{kq\gamma}{r'^2} \qquad \text{(along } z'\text{)}$$

Field from a charge moving at constant velocity

1. No static charge distribution could produce this electric field
2. The integral of $\vec{E} \cdot d\vec{l}$ around closed paths is not zero ...
3. ...as it is in electrostatics
4. $\Longrightarrow$ Maxwell's equations imply a time-varying magnetic flux.
5. Associated with moving charge is not just $\vec{E}$, but also $\vec{B}$
6. $\vec{B}$ is just $\vec{E}$ viewed in relative motion



$v = 0$ $\quad v = 0.75c$ $\quad v = 0.9c$ $\quad v = 0.99c$

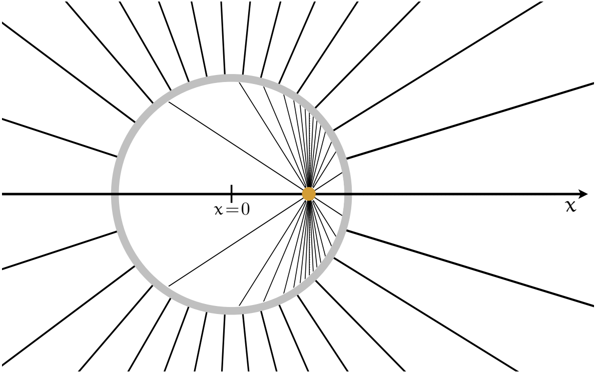
Outline

Fields of charges that start and stop

1. In free space, electromagnetic influences travel at c
2. Shake a charge here, one over there shakes a little later ...
3. Charge q initially at rest
4. At $t=0$ accelerated to a constant velocity v along the x
5. Assume constant acceleration a over time τ
6. Assume $\tau \ll$ the time scale we observe charge
7. What does the field look like surrounding the charge?

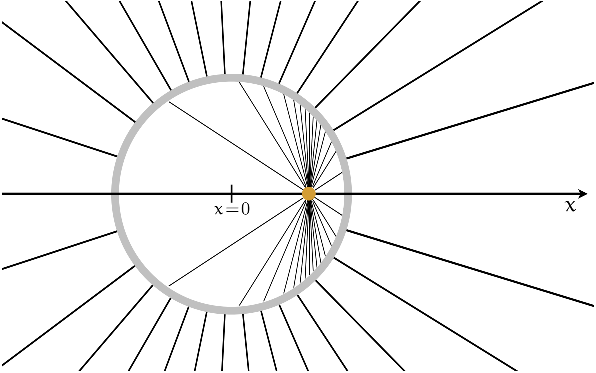
Fields of charges that start and stop

1. Consider observer at position r at T after acceleration begins
2. Has enough time has passed for the “news” to reach r?
3. If $r > cT$, then not enough time has passed.
4. Thus, charge still appears stationary! Field of point charge at rest.
5. For $r > cT$ *field still originates from charge's position at time* $t = 0$!



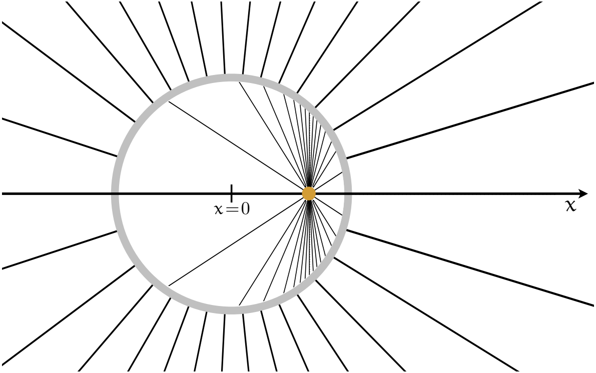
Fields of charges that start and stop

1. $r < c(T - \tau)$? Enough time has passed that the news to arrive.
2. For $r < c(T - \tau)$, charge is done accelerating, at constant v
3. Observers with $r < c(T - \tau)$ see the field of a moving point charge
4. See charge moved forward to $x_o = v(T - \tau) + \frac{1}{2}a\tau^2$
5. For $c(T - \tau)$ see the field of a moving point charge at x_o



Fields of charges that start and stop

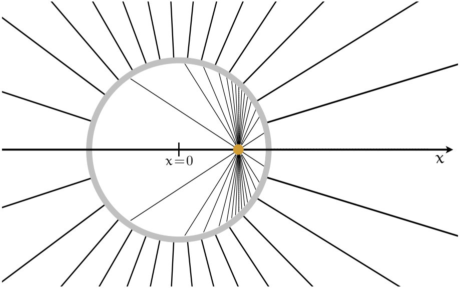
1. Spherical shell corresponding to accel. phase moves outward
2. Observers at progressively larger distances from the origin begin to see dramatic change in field
3. What happens inside the spherical shell?
4. Field lines cannot cross, number is fixed by q (Gauss' law)
5. *Field lines inside and outside shell must connect to each other in shell*



Fields of charges that start and stop

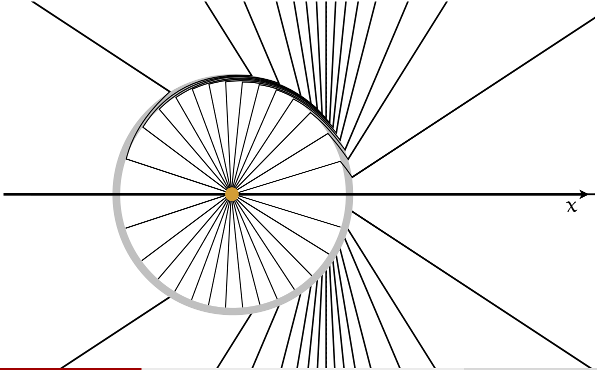
1. Connecting lines are not radially outward
2. This means field in shell has a *transverse* component now
3. As q accelerates, "sheds" part of its $\vec{E}$ field within shell, which travels outward at c.
4. Field in shell volume = energy carried away from charge
5. This energy is *electromagnetic radiation*

Charge is losing the energy contained in the electric field within the shell. If it is losing energy it must experience a force!



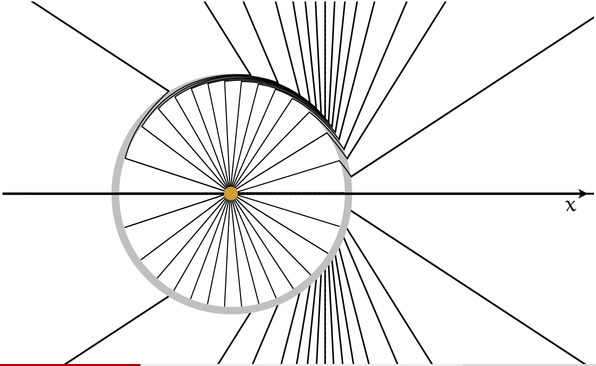
Fields of charges that start and stop

1. Consider a charge which suddenly *stops* instead
2. q moving with v until reaching the origin at $t=0$ and stops
3. For $r > ct$, news of deceleration has not been received ...
4. ... the field is that of a point charge in motion at v ...
5. ... emanating from a point vt past the origin on the x axis.
6. Within shell, enough time, field is point charge at rest at the origin.



Fields of charges that start and stop

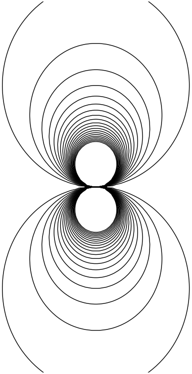
1. Inside shell representing deceleration period, lines connect
2. Precise shape depends on the details of the acceleration
3. Key: they are *transverse* with almost no radial component
4. Key: field within the shell propagates outward as a pulse.
5. Given that $\vec{E}$ is a function of time, there will also be $\vec{B}$ associated
6. Together $\vec{E}$ and $\vec{B}$ make up an electromagnetic pulse.



Fields of charges that start and stop

Contours of constant power for charge undergoing uniform acceleration along the horizontal axis.

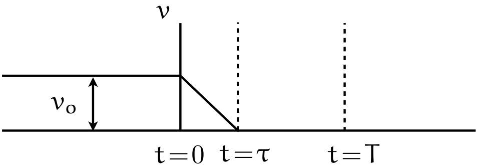
Next: the formula for the radiated power.



Outline

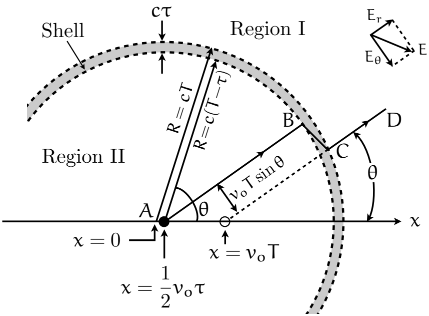
Radiation of accelerating charges

1. Charge q has been traveling at velocity v_o along the x axis
2. Suddenly at time $t=0$ at $x=0$ decelerates smoothly for time τ
3. (implying acceleration $a=v_o/\tau$)
4. Comes to rest as shown below



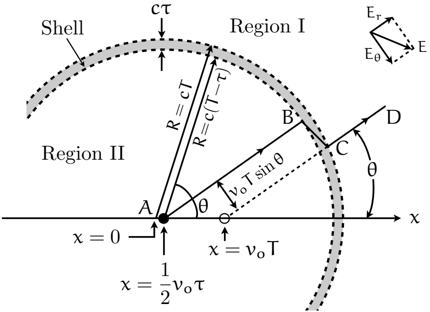
Radiation of accelerating charges

1. During deceleration: q moves $x = \frac{1}{2}v_o\tau$ before stopping
2. If $v_o \ll c$, x is tiny compared to relevant distances (e.g, $c\tau$)
3. At $t = T \gg \tau$, what does the field look like?
4. Observer at d doesn't know charge stopped until $\delta t = d/c$ later!
5. For $R > cT$ cannot know that the charge has stopped yet
6. For $R < c(T-\tau)$ already see the charge as stationary.
7. Within shell at $c(T-\tau) < R < cT$? See the charge decelerating



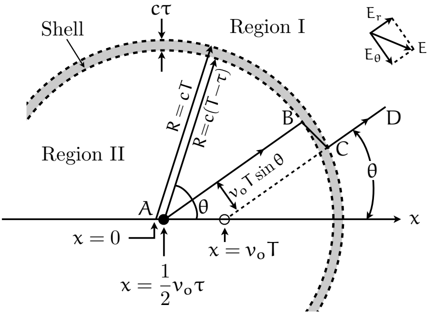
Region I

1. Outside shell, $R > cT$, see charge moving at constant velocity v_o!
2. Appears that nothing has changed
3. As though q is still moving at v_o, and at position $x = v_oT$ at T.
4. Field appears to emanate *where the charge would be*
5. Lines compressed along the axis $\perp$ to v ($\overline{CD}$)



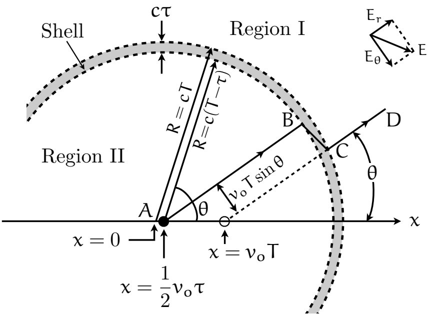
Region II

1. Enough time to see deceleration
2. For $R < c(T-\tau)$, see charge at rest at $x = \frac{1}{2}v_o\tau$
3. Field lines emanate radially from the charge's position ($\overline{AB}$).



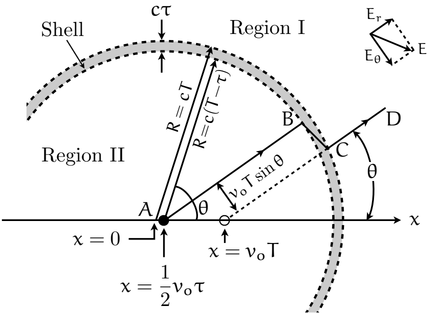
The Shell

1. Between I and II, $c(T-\tau) < R < cT$
2. See the charge in the midst of its deceleration
3. Field in this region? Must connect I and II ($\overline{BC}$)
4. This field is the radiation



Radiation of accelerating charges

1. In shell: lines like $\overline{BC}$.
2. Has radial and tangential components.
3. In II, see stationary point charge, purely radial field.
4. Gauss' law: flux in and out of shell same, determined by q alone
5. Flux only non-zero due to radial component
6. *Radial portion of the field cannot change when going from region II to the shell.*



Radiation of accelerating charges

In II the radial component of the field is that of a point charge, and it must be the same inside the shell:

$$E_r = \frac{q}{4\pi\epsilon_o R^2} = \frac{q}{4\pi\epsilon_o c^2 T^2}$$

From geometry of figure: $\tan\theta = \frac{E_r}{E_\theta} = \frac{c\tau}{v_o T \sin\theta}$

This gives us the tangential portion of the field:

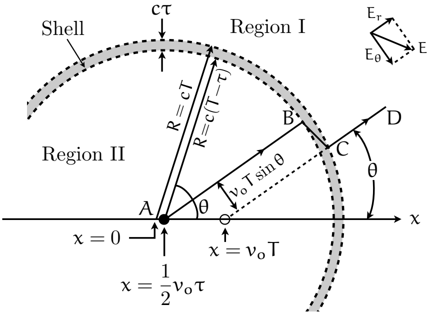
$$E_\theta = \frac{E_r}{\tan\theta} = E_r \frac{v_o T \sin\theta}{c\tau} = \frac{q v_o \sin\theta}{4\pi\epsilon_o c^3 T\tau}$$

Note $a = v_o/\tau$ and $R{=}cT$:

$$E_\theta = \frac{q a \sin\theta}{4\pi\epsilon_o c^2 R}$$

Radiation of accelerating charges

1. Tangential field E_θ goes as $1/R$, not $1/R^2$!
2. As R (or t) increases, E_θ will overcome E_r due to slower decay
3. In II we have the field of a point charge at constant velocity, which has both radial and tangential components.
4. In I, we have the purely radial field of a stationary point charge.
5. During deceleration, tangential component of $\vec{E}$ 'lost' as radiation, emanates outward from the charge at c in a shell of width $c\tau$.



Radiation of accelerating charges

End of lecture 5.

Radiation of accelerating charges

Radiation of accelerating charges

How much energy is lost during acceleration by radiation? Recall tangential component inside spherical shell width τ:

$$E_\theta = \frac{q v_o \sin\theta}{4\pi\epsilon_o c^3 T\tau}$$

1. What is the energy density of E_θ in the shell?
2. In general, $u = \frac{\text{energy}}{\text{volume}} = \frac{1}{2}\epsilon_o E^2$

$$u_\theta = \frac{1}{2}\epsilon_o E_\theta^2 = \frac{q^2 a^2 \sin^2\theta}{32\pi^2\epsilon_o c^4 R^2}$$

Volume of shell is $4\pi R^2 c\tau \ldots$

Radiation of accelerating charges

Multiply energy density by volume $4\pi R^2 c\tau$ to get radiated energy:

$$U_\theta = u_\theta V = \frac{q^2 a^2 \tau \sin^2\theta}{8\pi\epsilon_o c^3}$$

1. But there is also a magnetic field, and we know $u_E = u_B$.
2. Just double the result to account for B.
3. Convenient: average over all angles.
4. $\langle \sin^2\theta \rangle = \frac{2}{3}$ over a sphere

$$\langle U_\theta \rangle = \frac{q^2 a^2 \tau}{6\pi\epsilon_o c^3}$$

Radiation of accelerating charges

1. This is the *entire* energy emitted over the acceleration phase
2. Better is the *power* - energy per unit time.
3. $P = U/\tau$, so

$$P_{rad} = \frac{\langle U_\theta \rangle}{\tau} = \frac{q^2 a^2}{6\pi\epsilon_o c^3}$$

1. Total power, E and B fields
2. If you want angle-resolved, skip averaging step:

$$P_{rad} = \frac{q^2 a^2 \sin^2\theta}{4\pi\epsilon_o c^3}$$

Radiation of accelerating charges

$$P_{\text{rad}} = \frac{q^2 a^2}{6\pi\epsilon_o c^3}$$

1. Note power goes as the *square* of acceleration - sign is irrelevant
2. Also square of charge, sign of charge also irrelevant.
3. Independent of reference frame.
4. This is the *Larmor* equation.
5. Next: what about a charge in simple harmonic motion?

Outline

Charges in simple harmonic motion

1. For SHM, we know $x(t) = x_o \cos \omega_o t$
2. Choose phase of zero for convenience, changes nothing.
3. Thus $a = -\omega_o^2 x = -\omega_o^2 x_o \cos \omega_o t$
4. Can we just plug this in?
5. Yes, but more useful: power averaged over one period of motion.
6. Note $\langle \cos^2 \omega t \rangle = \frac{1}{2}$.

$$\langle a^2 \rangle = \langle -\omega_o^4 x_o^2 \cos^2 \omega_o t \rangle = -\omega_o^4 x_o^2 \langle \cos^2 \omega_o t \rangle = -\frac{1}{2} \omega_o^4 x_o^2$$

Charges in simple harmonic motion

1. Now we can use the Larmor formula.
2. Total emitted power per cycle:

$$P = \frac{q^2 \omega_o^4 x_o^2}{12\pi\epsilon_o c^3}$$

1. Goes as square of amplitude (as does energy in general for SHO).
2. Goes as ω_o^4 - much more severe as frequency increases.
3. Since charge loses energy, amplitude decays!
4. Physically: emits radiation at resonance frequency ω_o.

Charges in simple harmonic motion

1. Loss of energy means the oscillator is *damped*.
2. Will cover this in PH301/2 extensively.
3. Rate of loss = Q factor, "quality".

$$Q = 2\pi \frac{\text{total energy of oscillator}}{\text{rate of energy loss per radian}} = \omega_o \frac{\text{energy stored}}{\text{power loss}}$$
$$= \omega_o \frac{\mathcal{E}}{d\mathcal{E}/d\theta} = \frac{\omega_o \mathcal{E}}{P}$$

1. Equivalently, $Q = \omega/\Delta\omega$, $\Delta\omega$ is width of resonance

Charges in simple harmonic motion

Since $P = d\mathcal{E}/dt$, $\mathcal{E}$ is energy, can say:

$$P = -\frac{d\mathcal{E}}{dt} = -\frac{\omega\mathcal{E}}{Q}$$

$$\implies \qquad \mathcal{E} = \mathcal{E}_o e^{-\omega_o t/Q}$$

1. $\mathcal{E}_o$ is energy at $t = 0$.
2. Energy decays exponentially, time constant Q/ω_o.
3. Math is the same as an RLC circuit.
4. Great, but what is Q?

Charges in simple harmonic motion

The average energy for SHO is always half kinetic and half potential:

$$\langle \mathcal{E} \rangle = \frac{1}{2} m\omega_o^2 x_o^2$$

Vibrating at its natural frequency ω_o, this gives us

$$\frac{1}{Q} = \frac{P}{\omega_o \mathcal{E}} = \frac{q^2 \omega_o^4 x_o^2}{12\pi\epsilon_o c^3} \left(\frac{1}{\frac{1}{2} m\omega_o^2 x_o^2} \right) \left(\frac{1}{\omega_o} \right) = \frac{q^2 \omega_o}{6\pi\epsilon_o c^3 m}$$

In terms of wavelength $\lambda_o = 2\pi c / \omega_o$,

$$\frac{1}{Q} = \frac{q^2}{3\epsilon_o m c^2 \lambda_o} = \left(\frac{q^2}{4\pi\epsilon_o m c^2} \right) \left(\frac{1}{\lambda_o} \right) \left(\frac{4\pi}{3} \right) = \frac{4\pi}{3} \frac{r_e}{\lambda_o}$$

Charges in simple harmonic motion

1. $r_e = q^2/4\pi\epsilon_o mc^2$ has units of length
2. known as the *classical electron radius*
3. Q depends only on the ratio r_e/λ, so is Q dimensionless
4. For $q = e$, $r_e \approx 2.8 \times 10^{-15}$ m
5. The electron is, as far as we can tell, a point particle.
6. *If* the electron were a uniform sphere of charge, r_e is roughly the size an electron would need to be for its rest energy to be completely due to electrostatic potential energy
7. Know this to be incorrect now.

Charges in simple harmonic motion

1. What is the Q value for a typical atom?
2. Na discharge lamp, $\lambda = 600\,\text{nm}$ (yellow)
3. $Q = \frac{4\pi r_e}{\lambda_o} = \frac{3\epsilon_o m c^2 \lambda_o}{e^2} \sim 10^8$
4. Atom will oscillate $\sim 10^8$ radians or $\sim 10^7$ cycles before the energy is reduced by a factor $1/e \approx 1/2.718 \approx 0.37$.
5. Compare: $Q \sim 1000$ good RLC, 10^4 quartz, 10^6 precision circuit
6. $\lambda = 600\,\text{nm}$ implies a period of $\sim 10^{-15}\,\text{s}$
7. Takes about $10^{-8}\,\text{s}$ for the energy to decay by a factor of $1/e$.

Charges in simple harmonic motion

1. Q factor can be related to the *damping constant* γ
2. γ is the coefficient of the 'viscous' force proportional to velocity
3. $m\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + kx = 0$
4. Damping and Q relate via $\frac{1}{Q} = 2\gamma$
5. Thus $\gamma = \frac{q^2\omega_o}{12\pi\epsilon_o c^3 m}$
6. For a series RLC circuit, $\gamma = (R/2)\sqrt{C/L}$.
7. Knowledge of the damping factor or Q factor also allow us to find the width of the resonance $\Delta\omega$, since $\Delta\omega = \omega_o/Q$.
8. More useful is linewidth $\Delta\lambda$

Charges in simple harmonic motion

1. Since $\lambda_o = c/f = 2\pi c/\omega_o \ldots$
2. Relative linewidth: propagation of variation/uncertainty
3. $|\Delta\lambda| = \frac{d\lambda}{d\omega}(\Delta\omega) = (2\pi c/\omega^2)\Delta\omega$ and $\Delta\omega = \omega_o/Q$

$$\Delta\lambda = \frac{2\pi c \Delta\omega}{\omega_o^2} = \frac{2\pi c}{Q\omega_o} = \frac{e^2}{3\epsilon_o mc^2} = \frac{e^2}{4\pi\epsilon_o mc^2}\frac{4\pi}{3} = \frac{4\pi r_e}{3}$$

1. Again, relates to characteristic length r_e. For Na, $\Delta\lambda \sim 10^{-14}$ m.
2. Relative linewidth (the "sharpness" of the line) is then

$$\frac{\Delta\lambda}{\lambda_o} = \frac{4\pi r_e}{3\lambda_o} \sim 10^{-8}$$

Back to spectral lines extensively when we have a good atomic model.

Charges in simple harmonic motion

1. Can do a similar analysis for orbiting charges
2. Due to radiation loss, orbit decays exponentially
3. For a model hydrogen atom, decay time is $\sim 10^{-11}$ s!
4. Orbiting electron model is not workable.
5. Let's push our radiation model and see where it fails …

Outline

Radiation reaction force

1. If the charge is accelerating, it is losing energy.
2. If an oscillating charge loses energy, amplitude decays.
3. Radiation amounts to a *damping force*.
4. Start from Larmor:

$$P = \frac{e^2 a^2}{6\pi\epsilon_o c^3}$$

Relate power to force and velocity:

$$P = \int \vec{F} \cdot \vec{v}\, dt$$

Radiation reaction force

1. Consider power emitted by our oscillator from time t_1 to time t_2
2. Let this be exactly one period: $t_1 - t_2 = T = 1/f_o$.
3. Point charge of mass m and charge e, natural resonance frequency $\omega_o = 2\pi f_o$.
4. Like charge q, m on a spring k.
5. Conservation of energy: power radiated must equal the mechanical power:

$$0 = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} \, dt + \int_{t_1}^{t_2} P \, dt \qquad \text{or} \qquad \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} \, dt = -\int_{t_1}^{t_2} P \, dt$$

Restricting to non-relativistic velocities ($v \ll c$) for simplicity

Radiation reaction force

Note $a = dv/dt$:

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = -\int_{t_1}^{t_2} P\, dt = -\int_{t_1}^{t_2} \frac{e^2 a^2}{6\pi\epsilon_o c^3}\, dt = -\int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3} \frac{d\vec{v}}{dt} \cdot \frac{d\vec{v}}{dt}\, dt$$

We can integrate by parts:

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d\vec{v}}{dt} \cdot \vec{v} \Bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v}\, dt$$

We integrate over a full cycle, the first term vanishes – $\frac{d\vec{v}}{dt} \cdot \vec{v}$ is the same at each limit.

Radiation reaction force

$$\Longrightarrow \qquad \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \int_{t_1}^{t_2} \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} \cdot \vec{v}\, dt$$

Since $P = \int \vec{F} \cdot \vec{v}\, dt$ we can identify

$$\vec{F} = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^2\vec{v}}{dt^2} = \frac{e^2}{6\pi\epsilon_o c^3} \frac{d^3\vec{x}}{dt^3}$$

1. Effective damping force acting the oscillating charge due to the fact that it is radiating.
2. Known as the *Abraham-Lorentz force.*
3. Emitted radiation carries away momentum, charge must be pushed in opposite direction.
4. Problem: the charge is exerting a force on itself?!?!
5. Only resolved with quantum electrodynamics (QED).

Outline

Equation of motion

1. Oscillating charge experiences damping due to the radiation
2. Damping force similar to viscous fluid drag for a mechanical oscillator.
3. Without damping, for simple harmonic motion:

$$F = ma = -kx \qquad \text{or} \qquad m\frac{d^2x}{dt^2} = -kx = -m\omega_o^2 x$$

Now include radiation reaction force derived above. Acts as the same direction as the restoring force:

$$\begin{aligned} F = m\frac{d^2x}{dt^2} &= -m\omega_o^2 x - \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3} \qquad \text{or} \\ 0 = m\frac{d^2x}{dt^2} &+ m\omega_o^2 x + \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3} \end{aligned}$$

Equation of motion

1. Amplitude of oscillation will decay with time
2. Not interested in the isolated case of a single oscillator.
3. We want oscillator interacting with an electric field.
4. This means a *driven* oscillator.
5. Simplest: charge exposed to a monochromatic electric field, i.e., an electric field which varies sinusoidally with time with a single frequency $\omega = 2\pi f$:

$$|\vec{E}| = E_o \cos \omega t$$

In general $\omega \neq \omega_o$, frequency of the driving field not the same as the resonance frequency of the oscillating charge.

Equation of motion

1. Time-varying $\vec{E}$ produces a time-varying force $e|\vec{E}|$ on charge
2. This is the driving force for our oscillator.
3. Adding this driving force to our already-damped oscillator:

$$m\frac{d^2x}{dt^2} + \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3} + m\omega_o^2 x = eE_o \cos\omega t$$

1. Ugly third derivative. Will need to approximate.
2. Want it to look like a normal damped, driven oscillator.

Equation of motion

1. In most cases, radiation resistance force is small compared to the restoring force
2. This means small/light damping.
3. Then the acceleration is *approximately* the same as it is without damping, or $a \sim \omega_o^2 x$.
4. If this is the case,

$$\frac{d^2x}{dt^2} \sim \omega_o^2 x \qquad \text{so} \qquad \frac{d^3x}{dt^3} \sim \frac{d}{dt}\left(\omega_o^2 x\right) = \omega_o^2 \frac{dx}{dt}$$

Using this, we get something like a normal damped, driven oscillator.

Equation of motion

Substituting:

$$m\frac{d^2x}{dt^2} + \frac{e^2\omega_o^2}{6\pi\epsilon_o c^3}\frac{dx}{dt} + m\omega_o^2 x = eE_o\cos\omega t \qquad \text{(cancel } m\text{)}$$

$$\frac{d^2x}{dt^2} + \frac{e^2\omega_o^2}{6\pi\epsilon_o mc^3}\frac{dx}{dt} + \omega_o^2 x = \left(\frac{eE_o}{m}\right)\cos\omega t$$

Define a "damping constant" γ

$$\gamma = \frac{e^2\omega_o}{12\pi\epsilon_o mc^3}$$

$$\Longrightarrow \qquad \frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{eE_o}{m}\cos\omega t$$

Equation of motion

$$\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{eE_o}{m}\cos\omega t$$

Compare to series RLC circuit:

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \omega_o^2 I = \frac{\omega V_o}{L}\cos\omega t$$

1. Driven oscillator with a damping proportional to velocity.
2. Same as an LC resonant circuit with resistance included.
3. We know the solution to this equation:

$$x(t) = A\cos\left(\omega t + \varphi\right)$$

Just need to figure out what A and φ are ...

Equation of motion

Analogies:

	Series RLC	**Parallel RLC**	**Mechanical**
restoring	inverse capacitance $1/C$	inverse inductance $1/L$	spring constant k
"mass"	inductance L	capacitance C	mass m
friction	R	$1/R$	damping coefficient c
damping γ	$\frac{1}{2}R\sqrt{C/L}=\frac{1}{2}RC\omega_o=R/2L\omega_0$	$\frac{1}{2R}\sqrt{L/C}=\frac{1}{2R}L\omega_o=1/2RC\omega_o$	c/m
ω_o	$\sqrt{1/LC}$	$\sqrt{1/LC}$	$\sqrt{k/m}$
$Q=1/2\gamma$	$\frac{1}{R}\sqrt{L/C}$	$R\sqrt{C/L}=RC\omega_o=R/L\omega_o$	$m/2c$

Equation of motion

$$\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{eE_o}{m}\cos\omega t$$

1. Finding a steady-state solution? (Forget transients.)
2. Complex exponentials makes it easy.
3. Presume a solution like $x(t) = Ae^{i(\omega t+\varphi)}$, rest is algebra

$$A(\omega) = \frac{eE_o/m}{\sqrt{(\omega_o^2-\omega^2)^2 + (2\gamma\omega\omega_o)^2}}$$

$$\varphi = \tan^{-1}\left(\frac{2\omega\omega_o\gamma}{\omega^2-\omega_o^2}\right)$$

1. Resonance frequency is where $A(\omega)$ is maximal:
2. $\omega_r = \omega_o\sqrt{1-2\gamma^2}$
3. Small damping, reduces to $\omega_r \approx \omega_o(1-\gamma^2) \approx \omega_o$.

Equation of motion

$$A(\omega) = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \qquad \varphi = \tan^{-1}\left(\frac{2\omega\omega_o\gamma}{\omega^2 - \omega_o^2}\right)$$

1. From the phase equation, for low driving frequencies, $\omega < \omega_o$, the phase angle is small and the charge will oscillate in sync with the driving field.
2. When $\omega > \omega_o$, displacement is in the opposite direction from the driving force, 180° degrees out of phase with the field.
3. From amplitude equation: amplitude strongly decreases above ω_o, and more gradually below ω_o.
4. Sharp peak where the driving frequency matches the oscillator's resonance frequency, $\omega = \omega_r$.

Equation of motion

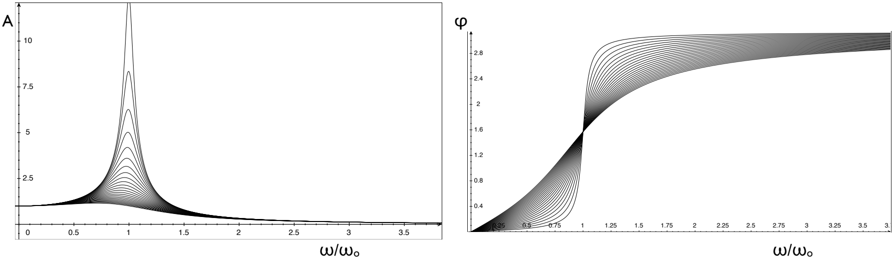


Figure: **(left)** Relative amplitude of oscillation versus driving frequency with γ ranging from 0.04 (top curve) to 0.5 (bottom curve) in steps of 0.02. The linewidth of the resonance curve is $\omega_o/2Q$. **(right)** Phase in radians versus driving frequency with γ ranging from 0.04 (sharpest curve) to 0.5 (smoothest curve) in steps of 0.02.

Equation of motion

1. Given the amplitude, potential energy: $U = \frac{1}{2} m\omega_o^2 A^2$.
2. Averaged over a cycle, the K and U of the oscillator are the same.
3. Total average energy is $m\omega_o^2 A^2$ … use A as we derived.
4. Check: reduce to no damping: $\gamma \to 0$, which gives

$$A(\omega) = \frac{eE_o}{m\left(\omega_o^2 - \omega^2\right)} \qquad (\gamma \to 0)$$

1. Just what we expect for a driven oscillator without damping.
2. Remove the periodicity of the driving force ($\omega \to 0$)? Free oscillator in a static electric field:

$$A = \frac{eE_o}{m\omega_o^2} \qquad (\omega \to 0)$$

Same result one gets from a force balance, $m\omega_o^2 A = kA = eE$.

Equation of motion

1. What have we learned over all?
2. Our charged oscillator is driven by a periodic electric field, and this field 'feeds' energy into the oscillator, which is in turn drained away by radiation damping.
3. The charge absorbs energy from the electric field, and reemits it as radiation at the same frequency.
4. This leads to a steady-state equilibrium, in which the energy gained from the field balances the energy lost by radiation.
5. More importantly: building up a model of the interaction of radiation and matter.

Next: scattering of light, thermal radiation.

Or ... why is the sky blue? Why do hot objects glow?

Equation of motion

End of lecture 6.

Today: leading up to thermal radiation

Practical matters

Exam next Wednesday in here. Coverage: most of Krane Ch. 2

- Length contraction
- Time dilation
- Lorentz transformations
- Dynamics
- A single question involving the Larmor formula

Exam rules

- Phones off. Only dumb calculators with no communication.
- Formula sheet provided, will give any not on the sheet.
- Show your work on all problems, no multiple choice.
- Right work more important than right answer.
- Likely: 5 problems given, solve any 4.

Study? Read the book. Old HW. Old exams.
`http://pleclair.ua.edu/PH253/`

Equation of motion

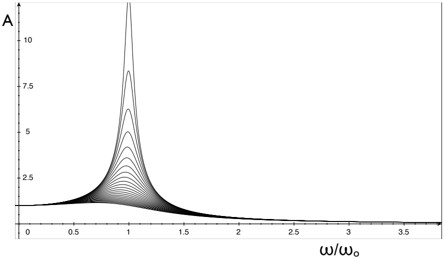
1. What have we learned over all?
2. Accelerating charges radiate energy.
3. Power is $P = \frac{e^2 a^2}{6\pi\epsilon_o c^3}$
4. This results in a *resistive force* $F_{rad} = \frac{e^2}{6\pi\epsilon_o c^3}\frac{d^3x}{dt^3}$
5. For a bound charge oscillating at its resonance ω_o, results in *damping*.
6. If the charge is also driven by an external field at ω, resonant excitation when $\omega \approx \omega_o$
7. If damping is small,

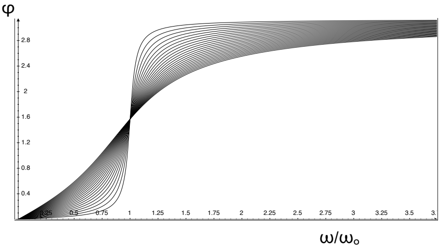
$$A(\omega) = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \qquad \varphi = \tan^{-1}\left(\frac{2\omega\omega_o\gamma}{\omega^2 - \omega_o^2}\right)$$

Equation of motion

1. What do you mean "small"?
2. Take amplitude of vibration $A \sim 0.1\,\mathrm{nm}$ (very large for an atom!)
3. Incident red light ($\omega_o/2\pi = f_o \sim 5 \times 10^{14}$ Hz)
4. Max acceleration of $\omega_o^2 A$ over a time of $1/f_o \approx 10^{-15}$ s
5. Gives a reaction force in the 10^{-18} N range.
6. HCl molecule, force constant $k \sim 500\,\mathrm{N/m}$, displacement of $0.1\,\mathrm{nm}$ gives restoring force $\sim 10^{-8}$ N
7. Electron in H atom - same order
8. Factor of 10^{10} greater than damping . . . small indeed.
9. Consistent with $Q \sim 10^8$, another way to say dissipation is small.

Equation of motion





1. Charge excited to maximum amplitude when $\omega \approx \omega_o$
2. $\omega \ll \omega_o$, in phase oscillation.
3. $\omega \gg \omega_o$, out of phase oscillation.
4. Smaller damping, more narrow resonance, larger peak.
5. Another view: this is how electrons bonded to atoms *scatter* light!

Outline

Setup

1. Instead of a single oscillating charge, how about many?
2. Expect both constructive and destructive interference of emitted radiation.
3. *Random* collection of atoms with oscillating charges? No net constructive or destructive interference
4. Total intensity is just the sum of the individual atoms.
5. Even in a regular crystal, random thermal motion ...
6. $\Longrightarrow$ Assume that all the atoms incoherently emit radiation
7. $\Longrightarrow$ Use properties of a single atom and multiply by the number of atoms
8. Setup: incident light in a single direction falling on an atom, and being reemitted over a range of angles = *scattering*

Setup

1. Incident EM wave strikes at atom, $\vec{E} = \vec{E}_o e^{i\omega t}$
2. Electron(s) will feel a periodic force $q\vec{E}$ and begin to vibrate
3. Electron accelerates, re-radiates some of the energy it received
4. This is *scattering* of light, and our driven harmonic oscillator
5. Have the amplitude and phase, so:

$$x(t) = \frac{eE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \cos(\omega t + \varphi)$$

Since damping is very small, adds little new physics here.

Ignore it for now.

Scattered light intensity

1. For the moment neglect damping ($\gamma \to 0$)
2. May be several different resonance frequencies, but just worry about one.
3. No damping:

$$x(t) = \frac{eE_o \cos \omega t}{m\left(\omega_o^2 - \omega^2\right)}$$

1. Now find acceleration and get power (averaged over angles)

$$P = \frac{e^2 \omega_o^4 A^2}{12\pi\epsilon_o c^3} = \frac{e^2 \omega_o^4}{12\pi\epsilon_o c^3} \frac{e^2 E_o^2}{m^2\left(\omega_o^2 - \omega^2\right)^2}$$

$$= \left(\frac{1}{2}\epsilon_o E_o^2\right)\left(\frac{e^4}{6\pi\epsilon_o^2 c^3 m^2}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2}$$

Scattered light intensity

1. Substitute for the classical electron radius to simplify:

$$P = \left(\frac{1}{2}\epsilon_o E_o^2\right)\left(\frac{8\pi r_e^2 c}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2}$$

1. Key: scattered energy goes as the *square* of the field
2. Proportional to (time-averaged) energy density of the incident field $\frac{1}{2}\epsilon_o E_o^2$
3. Scattered radiation intensity is proportional to the incident radiation intensity.
4. Basically: the brighter the source, the brighter the scattered light!

Scattered light intensity

1. Another view:
2. Say we have light going through a surface of area σ.
3. How much radiant energy passes through it in a given time t?
4. Energy density, multiplied by area σ, multiplied by the distance light can travel in t: $\frac{1}{2}\epsilon_o E_o^2 \sigma c t$.
5. Rate energy passes through the surface (power transmission)?
6. Energy divided by t, or $P = \frac{1}{2}\epsilon_o c E_o^2 \sigma$.
7. Compare to what we had:

$$P = \frac{1}{2}\epsilon_o c E_o^2 \sigma = \left(\frac{1}{2}\epsilon_o E_o^2\right)\left(\frac{8\pi r_e^2 c}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2}$$

$$\Longrightarrow \qquad \sigma = \left(\frac{8\pi r_e^2}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2}$$

Scattered light intensity

$$\sigma = \left(\frac{8\pi r_e^2}{3}\right)\frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2}$$

1. Right-hand side does have units of area!
2. What is the meaning of this area?
3. Atom scatters portion of radiation, it falls on a certain area.
4. σ is that area - area of beam "blocked" by atom.
5. Identification of σ takes ratio of total energy scattered per second to the incident energy per square meter:

$$\sigma = \frac{P}{\frac{1}{2}\epsilon_o c E_o^2} = \frac{\text{total scattered energy per second}}{\text{incident energy per square meter per second}}$$

Scattered light intensity

$$\sigma = \frac{P}{\frac{1}{2}\epsilon_o c E_o^2} = \frac{\text{total scattered energy per second}}{\text{incident energy per square meter per second}}$$

1. σ is usually called a *scattering cross section*
2. Energy intercepted by an area σ of incident beam is the same as that scattered by the atom.
3. Measure of how much of the beam we would need to block to scatter away as much of the incident light as the atom does.
4. Thus, a sort of characteristic 'size' associated with scattering
5. Compare for different scattering mechanisms to gauge relative strength

Scattered light intensity

1. Not a real area to speak of – just oscillating point charges
2. *Effect* same as if we blocked an area σ of the beam
3. $P = \frac{1}{2}\epsilon_o c E_o^2 \sigma = \sigma c \langle u_E \rangle$, define

$$P_{\text{scattered}} = \sigma c \langle u_E \rangle = \sigma I_{\text{incident}}$$

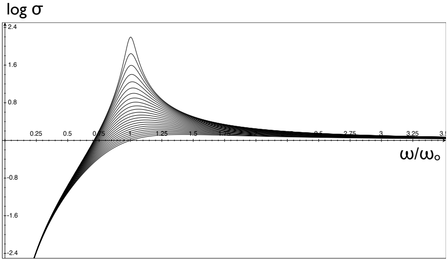
1. By definition, $I = c\langle u_{\text{field}} \rangle$.
2. I_{incident} is the *irradiance*, measure of radiation intensity.
3. Irradiance is the power flux per unit area (W/m^2), averaged over one period of oscillation
4. Scattered intensity proportional to the incident intensity
5. Brighter the source, the brighter the scattered light!

Scattered light intensity

1. Did not include radiation damping. Can go back and include it.
2. Clear that non-zero damping *reduces* the cross section
3. (i.e., the atoms are less effective scatterers)

$$\sigma = \left(\frac{8\pi r_e^2}{3}\right) \frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2 + \left(2\gamma\omega\omega_o\right)^2}$$

Scattering cross section highly dependent on ω, γ.



What conclusions can we draw?

$$P_{\text{scattered}} = \sigma I_{\text{incident}} \qquad \sigma = \left(\frac{8\pi r_e^2}{3}\right) \frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2 + \left(2\gamma\omega\omega_o\right)^2}$$

1. Scattering depends strongly on ω
2. Very large at the resonance ($\omega_o^2 - \omega^2$ denominator)
3. Incident radiation can most efficiently transfer its energy when frequencies match
4. At resonance the electron will most efficiently re-radiate
5. Numerator of the cross-section grows as ω^4 ...much larger above resonance than below
6. What is the resonance frequency for atmospheric gases?

What conclusions can we draw?

1. Atmospheric gases: resonances all in UV
2. Visible light is at much lower frequencies
3. We see cross section at frequencies *below* the resonance peak.
4. Higher frequency blue light is scattered more than lower frequency red
5. Look away from the sun: see light scattered the most = more blue
6. Toward sun at sunrise/sunset? See less scattered red light.
7. UV is absorbed even more strongly, which is a good thing.
8. Ozone is particularly good at absorbing ultraviolet light

What conclusions can we draw?

Mathematically: if $\omega \ll \omega_o$ and damping is negligible, σ reduces to

$$\sigma = \left(\frac{8\pi r_e^2}{3}\right) \frac{\omega^4}{\left(\omega_o^2 - \omega^2\right)^2} \approx \left(\frac{8\pi r_e^2}{3}\right) \frac{\omega^4}{\omega_o^4}$$

1. Cross section grows as ω^4 (or decreases as λ^{-4})
2. Higher frequency (smaller wavelength) radiation is scattered much more effectively
3. This is known as Rayleigh scattering
4. Left out some details, e.g. angular distribution, polarizability of the medium …

What did we learn?

1. Incident EM waves impinging on oscillating charges:
2. Close to resonance, charges absorb and reemit efficiently
3. This is scattering of light!
4. Strongly ω dependent
5. Explains blue sky and red sunrise/sunset
6. Scattering cross section = effective area of beam a particle blocks

Practical matters

Exam is Wednesday in here. Coverage: most of Krane Ch. 2

- Length contraction
- Time dilation
- Lorentz transformations
- Dynamics
- A single question involving the Larmor formula

Exam rules

- Phones off. Only dumb calculators with no communication.
- Formula sheet provided, will give any not on the sheet.
- Show your work on all problems, no multiple choice.
- Right work more important than right answer.
- Likely: 5 problems given, solve any 4.

Study? Read the book. Old HW. Old exams.

`http://pleclair.ua.edu/PH253/`

Outline

Thermal radiation

1. Know how to get radiation from oscillating charges
2. Know how they scatter incident radiation
3. Consider hot object (say, a gas in a perfectly black box)
4. Made up of many identical atoms, each has electrons that can oscillate and radiate.
5. Hot atoms in box acquire thermal energy, random motion induced.
6. Random = atoms have many different frequencies of oscillation …
7. …so any atom is exposed many frequencies at once, incoherently

Goal: energy emitted by a single atom in the box exposed to the radiation from all others.

Energy re-emitted by a single atomic oscillator driven by thermally-induced radiation

From this + thermo: spectrum of thermally-induced radiation

Thermal radiation

1. Already figured out the problem for a single incident frequency
2. Strongly peaked resonance since damping small
3. Means only driving frequencies that really matter are those close to the resonance frequency of the oscillator $\omega \approx \omega_r \approx \omega_o$
4. Only those frequencies give rise to a large amplitude of oscillation.
5. Using amplitude and total energy is kA^2:

$$U_{osc} = m\omega_o^2 A^2 = m\omega_o^2 \frac{e^2E^2/m^2}{\left(\omega_o^2 - \omega^2\right)^2 + 4\gamma^2\omega^2\omega_o^2}$$

Thermal radiation

$$U_{osc} = m\omega_o^2 A^2 = m\omega_o^2 \frac{e^2E^2/m^2}{(\omega_o^2 - \omega^2)^2 + 4\gamma^2\omega^2\omega_o^2}$$

1. If only frequencies with $\omega \approx \omega_o$ matter, approximation time!
2. First, a bit of factoring:

$$\left(\omega_o^2 - \omega^2\right)^2 = \left(\omega_o^2 - \omega^2\right)\left(\omega_o^2 - \omega^2\right) = (\omega_o - \omega)^2 (\omega_o + \omega)^2$$

If $\omega \approx \omega_o$, then $\omega_o + \omega \approx 2\omega_o$, and

$$\left(\omega_o^2 - \omega^2\right)^2 \approx 4\omega_o^2 (\omega_o - \omega)^2$$

Thermal radiation

$$\left(\omega_o^2 - \omega^2\right)^2 \approx 4\omega_o^2 \left(\omega_o - \omega\right)^2$$

Noting $\omega \approx \omega_o$ for the damping term in U

$$U_{osc} \approx \left(\frac{\omega_o^2}{m}\right) \frac{e^2E^2}{4\omega_o^2 \left(\omega_o - \omega\right)^2 + 4\gamma^2\omega_o^4} = \left(\frac{e^2E^2}{4m}\right) \frac{1}{\left(\omega - \omega_o\right)^2 + \gamma^2\omega_o^2}$$

1. For a single precise frequency of incident radiation ω
2. We want sum over all frequencies to find the total energy.
3. If U_{osc} is the energy of the oscillator at frequency ω …
4. …then $U_{osc}\, d\omega$ is the energy for $\omega \in [\omega, \omega + d\omega]$
5. Summing up frequency contributions = integrate $U(\omega)\, d\omega$

Thermal radiation

$U(\omega)$ is sharply peaked around ω_o, so limits of integral don't matter much. Take $0 \to \infty$ limits to make it easier.

$$U_{\text{osc,tot}} \approx \int_0^\infty \left(\frac{e^2E^2}{4m}\right) \frac{1}{(\omega-\omega_o)^2+(\gamma\omega_o)^2}\,d\omega$$

$$= -\frac{e^2E^2}{4m\gamma\omega_o}\tan^{-1}\left(\frac{\omega-\omega_o}{\gamma}\right)\Bigg|_0^\infty = \frac{\pi e^2E^2}{8m\gamma\omega_o}$$

1. Have missed one important detail: there are 2 polarizations
2. NBD, multiply by 2 as they are equivalent.

$$U_{\text{osc,tot}} = \frac{\pi e^2E^2}{4m\gamma\omega_o}$$

Thermal radiation

Recall definition of γ and factor:

$$U_{osc,tot} = \frac{\pi e^2 E^2}{4m\gamma\omega_o} = \left(\frac{1}{2}\epsilon_o E^2\right)\frac{6\pi^2 c^3}{\omega_o^2}$$

Term in brackets - energy density of the field. Rearrange

$$u_{field} = \frac{1}{2}\epsilon_o E^2 = U_{osc,tot}\frac{\omega_o^2}{6\pi^2 c^3}$$

1. Have a relationship between energy of a single oscillating charge and the energy of the field it is immersed in.
2. One dimensional so far, but other 2 are the same; multiply by 3

$$u_{field} = \frac{\omega_o^2}{2\pi^2 c^3}U_{osc,tot} = \frac{2f^2}{c^3}U_{osc,tot}$$

Thermal radiation

$$u_{\text{field}} = \frac{\omega_o^2}{2\pi^2 c^3} U_{\text{osc,tot}} = \frac{2f^2}{c^3} U_{\text{osc,tot}}$$

1. $u_{\text{field}}(\omega)\, d\omega$ is energy per unit volume at ω in $[\omega, \omega + d\omega]$
2. Key: means that if we can find the total energy of a given oscillator by some means . . .
3. . . . we automatically know the energy contained in the radiation field at a given frequency
4. Next: use thermodynamics!
5. Particle has thermal energy $\frac{1}{2}k_b T$ per degree of freedom.
6. This is where it goes hilariously wrong

Thermal radiation

1. Thermo: each oscillator has an average energy $\langle U_{\text{osc,tot}} \rangle = k_B T$ at a temperature T independent of the oscillator's frequency
2. (ignoring factors of 1/2 or 3/2)
3. If the oscillator's energy is purely thermal, we expect

$$\langle u_{\text{field}} \rangle = \frac{2f^2}{c^3} \langle U_{\text{osc,tot}} \rangle = \frac{2k_B T f^2}{c^3}$$

1. Irradiance ("intensity") is $I = c \langle u_{\text{field}} \rangle$

$$I = \frac{2k_B T f^2}{c^2}$$

Thermal radiation

$$I = \frac{2k_B T f^2}{c^2}$$

1. Awesome right? This is the famous Rayleigh-Jeans law.
2. Intensity scales with T and f^2. Think about that.
3. Agrees with experiments at low f.
4. Large f? Energy density should be *arbitrarily* large as frequency increases! Everything is white hot …
5. We should be bathing in X- and gamma-rays. We are not.
6. "Ultraviolet catastrophe" – theory behaves stupidly at high f
7. Model has gone horribly wrong somewhere. Find and fix.
8. Wrong by assuming that oscillators of any f get the same energy

Thermal radiation

Next time: we fix the model with Planck's hypothesis.