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PH 495/ECE 493 LeClair & Kung — Spring 2011

Problem Set 1 Solutions

1. *Hecht 2.17* The wavefunction of a transverse wave on a string is

$$\psi(x,t) = (30.0\,\text{cm})\cos\left[(6.28\,\text{rad/m})\,x - (20.0\,\text{rad/s})\,t\right] \tag{1}$$

Compute the frequency, wavelength, period, amplitude, phase velocity, and direction of motion.

Solution: The wavefunction has the form

$$\psi(x,t) = A\sin\left(kx - \omega t + \delta\right) \tag{2}$$

which lets us immediately identify

$$\omega = 2\pi f = 20\,\text{rad/s} \qquad \Longrightarrow \qquad f = \frac{2\pi}{\omega} = \frac{10}{\pi}\,\text{s}^{-1} \tag{3}$$

$$k = \frac{2\pi}{\lambda} = 6.28\,\text{rad/m} \qquad \Longrightarrow \qquad \lambda = \frac{2\pi}{k} \approx 1.0\,\text{m} \tag{4}$$

$$A = 30\,\text{cm} \tag{5}$$

$$v = \frac{\omega}{k} \approx 3.18\,\text{m/s} \tag{6}$$

Since the argument has the form $kx - \omega t$, the wave is traveling along the $+x$ direction.

2. *Hecht 2.18* Show that

$$\psi(x,t) = A\sin k\left(x - vt\right) \tag{7}$$

is a solution of the differential wave equation.

Solution: Taking the partial derivatives, we have:

$$\frac{\partial^2\psi}{\partial x^2} = -k^2\psi \tag{8}$$

$$\frac{\partial^2\psi}{\partial t^2} = -k^2v^2\psi \tag{9}$$

It is apparent that this satisfies the wave equation:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \tag{10}$$

3. *Hecht 2.31-2* Which of the following expressions correspond to traveling waves? For each of those, what is the speed of the wave? The quantities A, a, b, c are positive real constants.

$$\psi(x,t) = (ax - bt)^2 \tag{11}$$
$$\psi(x,t) = A\sin\left(ax^2 - bt^2\right) \tag{12}$$
$$\psi(x,t) = \frac{1}{ax^2 + b} \tag{13}$$
$$\psi(x,t) = A\sin 2\pi\left(\frac{x}{a} + \frac{t}{b}\right) \tag{14}$$

Solution: In order to be a traveling wave, the wavefunction must take the form $f(\alpha x \pm \beta t)$. Thus, only the first and last functions are traveling waves. The velocity is the ratio of the coefficient of the time term to the spatial term, $v = -\beta/\alpha$, and the sign of the time term tells us the direction. Thus for the first the velocity is $v = b/a$ in the $+x$ direction, and for the last $v = a/b$ along $-x$

4. *Hecht 2.38* Show that the imaginary part of a complex number z is given by

$$\frac{z - z^*}{2i} \tag{15}$$

Solution: Let $z = x + iy$ without loss of generality. Then $z^* = x - iy$, and

$$z - z^* = (x + iy) - (x - iy) = 2iy \tag{16}$$
$$\frac{z - z^*}{2i} = \frac{2iy}{2i} = y. \tag{17}$$

QED.

5. *Hecht 2.40* Show that the functions

$$\psi(x,y,z,t) = f(\alpha x + \beta y + \gamma z - vt) \tag{18}$$
$$\phi(x,y,z,t) = g(\alpha x + \beta y + \gamma z + vt) \tag{19}$$

which are plane waves of arbitrary form, satisfy the three-dimensional differential wave equation.

Solution: Let $u = \alpha x + \beta y + \gamma z \pm v t$ for both functions. Begin by taking partial derivatives with respect to x, y, z, and t. Using the chain rule,

$$\frac{\partial \psi}{\partial x} = \frac{d\psi}{du}\frac{\partial u}{\partial x} = \alpha f' \tag{20}$$

$$\frac{\partial^2 \psi}{\partial x} = \frac{d^2\psi}{du^2}\frac{\partial u}{\partial x} = \alpha^2 f'' \tag{21}$$

Similarly,

$$\frac{\partial^2 \psi}{\partial y^2} = \beta^2 f'' \tag{22}$$

$$\frac{\partial^2 \psi}{\partial z^2} = \gamma^2 f'' \tag{23}$$

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 f'' \tag{24}$$

The wave equation reads

$$\nabla^2 \psi = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} \tag{25}$$

Substitution yields

$$0 = \left(\alpha^2 + \beta^2 + \gamma^2\right) f'' - \frac{v^2}{v^2} f'' \tag{26}$$

$$\implies \qquad 1 = \alpha^2 + \beta^2 + \gamma^2 \tag{27}$$

Thus, the given wavefunctions are a solution to the wave equation if $\alpha^2+\beta^2+\gamma^2=1$.

6. *Hecht 3.4* The time average of some function $f(t)$ taken over an interval T is given by

$$\langle f(t) \rangle = \frac{1}{T}\int_{t}^{T+t} f(t')\, dt' \tag{28}$$

where t' is just a dummy variable of integration. If $\tau = 2\pi/\omega$ is the period of a harmonic function, show that

$$\langle \sin^2(kx - \omega t) \rangle = \frac{1}{2} \tag{29}$$

$$\langle \cos^2(kx - \omega t) \rangle = \frac{1}{2} \tag{30}$$

$$\langle \sin(kx - \omega t) \cos(kx - \omega t) \rangle = 0 \tag{31}$$

when $T = \tau$ and when $T \gg \tau$.

Solution: Starting with $\langle \sin^2(kx - \omega t) \rangle$, is is convenient to use a substitution

$$u = kx - \omega t' \tag{32}$$

$$du = -w\, dt' \tag{33}$$

Then we have

$$\begin{aligned}
\langle f(t) \rangle = \langle \sin^2(kx - \omega t) \rangle &= \frac{1}{T} \int_{t}^{T+t} \sin^2\left(kx - \omega t'\right) dt' = \frac{-1}{\omega T} \int_{t'=t}^{t'=T+t} \sin^2 u\, du \\
&= \frac{-1}{2\omega T} \left[u - \sin u \cos u\right] \Big|_{t'=t}^{t'=T+t} = \frac{-1}{2\omega T} \left[(kx - \omega t) - \sin(kx - \omega t) \cos(kx - \omega t)\right] \Big|_{t}^{T+t} \\
&= \frac{-1}{2\omega T} \left[-\omega T - \sin(kx - \omega t - \omega T) \cos(kx - \omega t - \omega T) + \sin(kx - \omega t) \cos(kx - \omega t)\right] \\
&= \frac{1}{2} + \frac{1}{2\omega T} \Big(\sin(kx - \omega t - \omega T) \cos(kx - \omega t - \omega T) - \sin(kx - \omega t) \cos(kx - \omega t) \Big)
\end{aligned} \tag{34}$$

For the specific limit of $T = \tau = \frac{2\pi}{\omega}$, we note that $\omega T = \omega \tau = 2\pi$. Thus,

$$\langle \sin^2(kx - \omega t) \rangle = \frac{1}{2} + \frac{1}{2\omega T} \Big(\sin(kx - \omega t - 2\pi) \cos(kx - \omega t - 2\pi) - \sin(kx - \omega t) \cos(kx - \omega t) \Big) \tag{35}$$

Since $\sin(\theta \pm 2\pi) = \sin\theta$ and $\cos(\theta \pm 2\pi) = \cos\theta$, the second term vanishes, we have

$$\langle \sin^2(kx - \omega t) \rangle = \frac{1}{2} \tag{36}$$

In the limit $T \gg \tau$, we notice that the second term in Eq. 34 goes as T in the denominator, while the sin and cos functions in the numerator are each at most 1. Thus, the entire second term goes as (const)/T. In the limit of large T (say $\mathsf{T} \to \infty$), this term vanishes.

$$\lim_{\mathsf{T}\to\infty} \langle f(t) \rangle = \lim_{\mathsf{T}\to\infty} \frac{1}{2} + \frac{\sin(kx - \omega t - 2\pi)\cos(kx - \omega t - 2\pi) - \sin(kx - \omega t)\cos(kx - \omega t)}{2\omega \mathsf{T}} = \frac{1}{2}$$

For the second part, all we need to do is notice that

$$\int \sin^2 u \, du = \frac{1}{2}u - \frac{1}{2}\sin u \cos u + C \tag{37}$$

$$\int \cos^2 u \, du = \frac{1}{2}u + \frac{1}{2}\sin u \cos u + C \tag{38}$$

Both integrals are the same, except for the change of sign of the second term. In both limits considered, the second term vanishes, so its sign is irrelevant.[i] Thus,

$$\langle \sin^2(kx - \omega t) \rangle = \langle \cos^2(kx - \omega t) \rangle = \frac{1}{2} \qquad \mathsf{T} \gg \tau, \mathsf{T} = \tau \tag{39}$$

Finally, we are left with

$$\langle f(t) \rangle = \langle \sin(kx - \omega t)\cos(kx - \omega t) \rangle \tag{40}$$

Using the same substitution above, we find

$$\begin{aligned}
\langle f(t) \rangle = \langle \sin(kx - \omega t)\cos(kx - \omega t) \rangle &= \frac{1}{\mathsf{T}} \int_{t}^{\mathsf{T}+t} \sin(kx - \omega t)\cos(kx - \omega t) \, dt' \\
&= \frac{-1}{\omega \mathsf{T}} \int_{t'=t}^{t'=\mathsf{T}+t} \sin u \cos u \, du = \frac{-1}{\omega \mathsf{T}} \left(\frac{1}{2}\sin^2 u \right)_{t'=t}^{t'=\mathsf{T}+t} = \frac{-1}{4\omega \mathsf{T}} \left(1 - \cos 2u \right)_{t'=t}^{t'=\mathsf{T}+t} \\
&= \frac{-1}{4\omega \mathsf{T}} \Big(-\cos(2kx - 2\omega t - 2\omega \mathsf{T}) + \cos(2kx - \omega t) \Big)
\end{aligned} \tag{41}$$

At the limit $\mathsf{T} = \tau$, since $\omega \mathsf{T} = \omega \tau = 2\pi$ the two terms in brackets cancel since $\cos\theta = \cos(\theta \pm 2\pi)$. In the limit $\mathsf{T} \gg \tau$, we note

$$\lim_{\mathsf{T}\to\infty} \frac{-\cos(2kx - 2\omega t - 2\omega \mathsf{T}) + \cos(2kx - \omega t)}{4\omega \mathsf{T}} = 0 \tag{42}$$

since the numerator can be at most 2 for any value of T. Thus,

$$\langle \sin(kx - \omega t)\cos(kx - \omega t) \rangle = 0 \qquad \mathsf{T} \gg \tau, \mathsf{T} = \tau \tag{43}$$

[i] If you like, repeat everything above with the appropriate signs flipped.

7. *Hecht 3.5* An electromagnetic wave is specified (in SI units) by the following function:

$$\vec{E} = \left(-6\,\hat{\imath} + 3\sqrt{5}\,\hat{\jmath}\right)\left(10^4\,\mathrm{V/m}\right) e^{i\left[\frac{1}{3}\left(\sqrt{5}x+2y\right)\pi\times 10^7 - 9.42\times 10^{15}\,t\right]} \tag{44}$$

Find **(a)** the direction along which the electric field oscillates, **(b)** the scalar value of the amplitude of the electric field, **(c)** the direction of propagation of the wave, **(d)** the propagation number and wavelength, **(e)** the frequency and angular frequency, and **(f)** the speed.

Solution: The field oscillates along the amplitude vector $-6\hat{\imath}+3\sqrt{5}\hat{\jmath}$. Normalizing to unit length, we have

$$\frac{-6\hat{\imath} + 3\sqrt{5}\hat{\jmath}}{\sqrt{6^2 + 9\cdot 5}} = -\frac{2}{3}\hat{\imath} + \frac{\sqrt{5}}{3}\hat{\jmath} \tag{45}$$

The scalar amplitude is

$$|\vec{E}| = \sqrt{\vec{E}\cdot\vec{E}} = 9\cdot 10^4\,\mathrm{V/m} \tag{46}$$

The form of the exponential is $e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}$. Given $\vec{r}=x\hat{x}+y\hat{y}$, by inspection we can find $\vec{k}$ and ω, and thus k, λ, f, and v.

$$\vec{k} = \left(\frac{\pi}{3}\cdot 10^7\right)\left(\sqrt{5}\hat{\imath} + 2\hat{\jmath}\right) \tag{47}$$

$$k = \sqrt{\vec{k}\cdot\vec{k}} = \pi\cdot 10^7\,\mathrm{m}^{-1} \tag{48}$$

$$\lambda = \frac{2\pi}{k} = 200\,\mathrm{nm} \tag{49}$$

$$\omega = 9.42\times 10^{15}\,\mathrm{rad/s} \tag{50}$$

$$f = \frac{\omega}{2\pi} = 1.5\cdot 10^{15}\,\mathrm{Hz} \tag{51}$$

$$v = \lambda f = 3\cdot 10^8\,\mathrm{m/s} = c \tag{52}$$

8. The equation for a driven damped oscillator is

$$\frac{d^2x}{dt^2} + 2\gamma\omega_o\frac{dx}{dt} + \omega_o^2 x = \frac{q}{m}E(t) \tag{53}$$

(a) Explain the significance of each term.
(b) Let $E = E_o e^{i\omega t}$ and $x = x_o e^{i(\omega t-\alpha)}$ where E_o and x_o are real quantities. Substitute into the

above expression and show that

$$x_o = \frac{qE_o/m}{\sqrt{(\omega_o^2 - \omega^2)^2 + (2\gamma\omega\omega_o)^2}} \tag{54}$$

(c) Derive an expression for the phase lag α, and sketch it as a function of ω, indicating ω_o on the sketch.

Solution: The significance of each term is probably more apparent if we re-arrange and multiply by mass:

$$m\frac{d^2x}{dt^2} = -m\omega_o^2 x - 2\gamma m\omega_o \frac{dx}{dt} + qE(t) \tag{55}$$

The term on the right is the net force on the oscillator. The first term on the left is the restoring force, the second the viscous damping term, and the last the driving force of the oscillator.

First, we find the derivatives of x, noting $i^2 = -1$:

$$\frac{dx}{dt} = i\omega x_o e^{i(\omega t - \alpha)} \tag{56}$$

$$\frac{d^2x}{dt^2} = -\omega^2 x_o e^{i(\omega t - \alpha)} \tag{57}$$

Substituting into the original equaiton,

$$\frac{q}{m}E_o e^{i\omega t} = -\omega^2 x_o e^{i(\omega t - \alpha)} + 2\gamma\omega_o i\omega x_o e^{i(\omega t - \alpha)} + \omega_o^2 x_o e^{i(\omega t - \alpha)} \tag{58}$$

$$\frac{q}{m}E_o e^{i\omega t} = e^{i(\omega t - \alpha)}\left(-\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o\right) \tag{59}$$

$$\frac{q}{m}E_o e^{i\omega t} = e^{i\omega t}e^{-i\alpha}\left(-\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o\right) \tag{60}$$

$$\frac{qE_o}{m}e^{i\alpha} = -\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o \tag{61}$$

To proceed, we use the Euler identity

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{62}$$

Giving

$$\frac{qE_o}{m}(\cos\alpha + i\sin\alpha) = -\omega^2 x_o + 2i\gamma\omega_o\omega x_o + \omega_o^2 x_o \tag{63}$$

We now have two separate equations if we equate the purely real and purely imaginary parts:

$$\frac{qE_o}{m}\cos\alpha = \omega_o^2 x_o - \omega^2 x_o \tag{64}$$
$$\frac{qE_o}{m}\sin\alpha = 2\gamma\omega\omega_o x_o \tag{65}$$

We can square both equations and add them together:

$$\frac{q^2E_o^2}{m^2}\left(\cos^2\alpha + \sin^2\alpha\right) = \left(\omega_o^2 - \omega^2\right)^2 x_o^2 + \left(2\gamma\omega\omega_o\right)^2 x_o^2 \tag{66}$$
$$x_o^2 = \frac{q^2E_o^2}{m^2}\frac{1}{\left(\omega_o^2 - \omega^2\right)^2 x_o^2 + \left(2\gamma\omega\omega_o\right)^2} \tag{67}$$
$$x_o = \frac{qE_o}{m}\frac{1}{\sqrt{\left(\omega_o^2 - \omega^2\right)^2 x_o^2 + \left(2\gamma\omega\omega_o\right)^2}} \tag{68}$$

This is the desired amplitude of vibration. Going back to the preceding two equations, we can divide the second equation by the first to find the phase angle:

$$\tan\alpha = \frac{2\gamma\omega\omega_o}{\omega_o^2 - \omega^2} \tag{69}$$

This is the same phase angle derived in the notes (modulo an overall sign due to the convention chosen), a sketch of phase angle versus frequency is provided there.

9. Calculate the divergence ($\vec{\nabla}\cdot$) and curl ($\vec{\nabla}\times$) for the following vector functions $\vec{F}(x,y,z)$. Then verify that the divergence of the curl is zero for each, i.e., $\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{F}\right)=0$.

$$x\hat{x} + (y+z)\,\hat{y} + (x+y+z)\,\hat{z}$$
$$f(x)\hat{x} + g(y)\hat{y} + h(z)\hat{z}$$
$$f(y,z)\hat{x} + g(z,x)\hat{y} + h(x,y)\hat{z}$$
$$(x+y+z)\,(x\hat{x} + y\hat{y} + z\hat{z})$$

Solution: All of the given functions have the form

$$\vec{F} = f(x,y,z)\hat{x} + g(x,y,z)\hat{y} + h(x,y,z)\hat{z} \tag{70}$$

so to save time we may first solve the general problem and then evaluate the specific cases. The divergence is simply

$$\nabla \cdot \vec{F} = \partial_x f + \partial_y g + \partial_z h \tag{71}$$

where $\partial_x \equiv \frac{\partial}{\partial x}$. The curl is

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial_x} & \frac{\partial}{\partial_y} & \frac{\partial}{\partial_z} \\ F_x & F_y & F_z \end{vmatrix} = \left(\partial_y h - \partial_z g\right)\hat{x} + \left(\partial_z f - \partial_x h\right)\hat{y} + \left(\partial_x g - \partial_y f\right)\hat{z} \tag{72}$$

And, finally, the divergence of the curl is zero since the order of partial derivatives does not matter (i.e., $\partial_x \partial_y f = \partial_y \partial_x f$):

$$\nabla \cdot \left(\nabla \times \vec{F}\right) = \partial_x \left(\partial_y h - \partial_z g\right) + \partial_y \left(\partial_z f - \partial_x h\right) + \partial_z \left(\partial_x g - \partial_y f\right) \tag{73}$$

$$= \partial_x \partial_y h - \partial_y \partial_x h + \partial_z \partial_x g - \partial_x \partial_z g + \partial_y \partial_z f - \partial_z \partial_y f = 0 \tag{74}$$

Thus, we don't need to specifically calculate $\nabla \cdot \left(\nabla \times \vec{F}\right)$ for each function. We do need to calculate the divergences and curls, however, which we can do quickly with the formulas above. For the first function, $f(x, y, z) = x$, $g(x, y, z) = y + z$, and $h(x, y, z) = x + y + z$:

$$\vec{F} = x\hat{x} + (y + z)\,\hat{y} + (x + y + z)\,\hat{z} : \tag{75}$$

$$\nabla \cdot \vec{F} = 1 + 1 + 1 = 3 \tag{76}$$

$$\nabla \times \vec{F} = \hat{x}\,(1 - 1) + \hat{y}\,(0 - 1) + \hat{z}\,(0 - 0) = -\hat{y} \tag{77}$$

For the second function, $f(x, y, z) \to f(x)$, $g(x, y, z) \to g(y)$, and $h(x, y, z) \to h(z)$:

$$\vec{F} = f(x)\hat{x} + g(y)\hat{y} + h(z)\hat{z} : \tag{78}$$

$$\nabla \cdot \vec{F} = \partial_x f + \partial_y g + \partial_z h \tag{79}$$

$$\nabla \times \vec{F} = 0 \tag{80}$$

For the third function, $f(x, y, z) \to f(y, z)$, $g(x, y, z) \to g(z, x)$, and $h(x, y, z) \to h(x, y)$:

$$\vec{F} = f(y, z)\hat{x} + g(z, x)\hat{y} + h(x, y)\hat{z} : \tag{81}$$

$$\nabla \cdot \vec{F} = \partial_x f + \partial_y g + \partial_z h = 0 \tag{82}$$

$$\nabla \times \vec{F} = \left(\partial_y h - \partial_z g\right)\hat{x} + \left(\partial_z f - \partial_x h\right)\hat{y} + \left(\partial_x g - \partial_y f\right)\hat{z} \tag{83}$$

For the last function, $f(x, y, z) \rightarrow x^2 + xy + xz$, $g(x, y, z) \rightarrow xy + y^2 + yz$, and $h(x, y, z) \rightarrow xz + yz + z^2$:

$$\vec{F} = (x + y + z)(x\hat{x} + y\hat{y} + z\hat{z}) : \tag{84}$$

$$\nabla \cdot \vec{F} = 2x + y + z + x + 2y + z + x + y + 2z = 4(x + y + z) \tag{85}$$

$$\nabla \times \vec{F} = \hat{x}(z - y) + \hat{y}(x - z) + \hat{z}(y - x) \tag{86}$$