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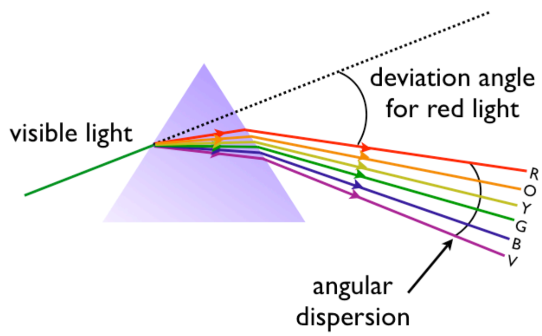
PH 495/ECE 493 LeClair & Kung | Spring 2011

Problem Set 2: Solutions

1. The index of refraction for **violet** light in silica flint glass is $n_{violet} = 1.66$, and for **red** light it is $n_{red} = 1.62$. In air, $n = 1$ for both colors of light.

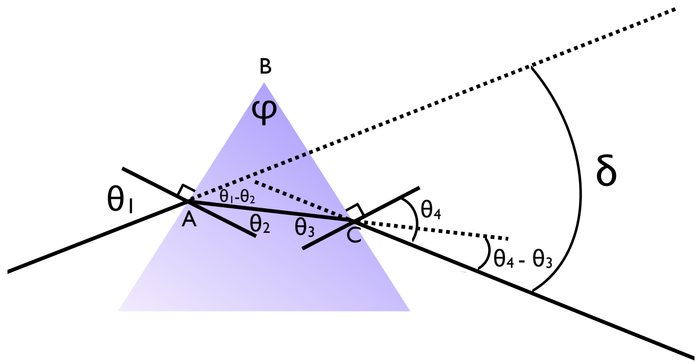
What is the **angular dispersion** of visible light (the angle between red and violet) passing through an equilateral triangle prism of silica flint glass, if the angle of incidence is 50°? The angle of incidence is that between the ray and a line *perpendicular* to the surface of the prism. Recall that all angles in an equilateral triangle are 60°.



Solution: What we need to do is find the deviation angle for both red and violet light in terms of the incident angle and refractive index of the prism. The angular dispersion is just the difference between the deviation angles for the two colors. First, let us define some of the geometry a bit better, referring to the figure below.
Let the angle of incidence be θ_1, and the refracted angle θ_2 at point A. The incident and refracted angles are defined with respect to a line *perpendicular* to the prism's surface. Similarly, when the light rays exit the prism, we will call the incident angle within the prism θ_3, and the refracted angle exiting the prism θ_4 at point C. If we call index of refraction of the prism n, and presume the surrounding material is just air with index of refraction 1.00, we can apply Snell's law at both interfaces:

$$n \sin \theta_2 = \sin \theta_1$$

$$n \sin \theta_3 = \sin \theta_4$$



Fair enough, but now we need to use some geometry to relate these four angles to each other, the deviation angle δ, and the prism's apex angle φ. Have a look at the triangle formed by points A, B, and C. All three angles in this triangle must add up to 180°. At point A, the angle between the prism face and the line $\overline{AC}$ is $\angle BAC = 90 - \theta_2$ - the line we drew to define θ_1 and θ_2 is by construction perpendicular to the prism's face, and thus makes a 90° angle with respect to the face. The angle $\angle BAC$ is all of that 90° angle, *minus* the refracted angle θ_2. Similarly, we can find $\angle BCA$ at point C. We know the apex angle of the prism is φ, and for an equilateral triangle, we must have $\varphi = 60°$

$$
\begin{aligned}
(90° - \theta_2) + (90° - \theta_3) + \varphi = 180° \\
\implies \varphi = \theta_2 + \theta_3 = 60°
\end{aligned}
$$

How do we find the deviation angle? Physically, the deviation angle is just how much in total the exit ray is "bent" relative to the incident ray. At the first interface, point A, the incident ray and reflected ray differ by an angle $\theta_1 - \theta_2$. At the second interface, point C, the ray inside the prism and the exit ray differ by an angle $\theta_4 - \theta_3$. These two differences *together* make up the total deviation - the deviation is nothing more than adding together the differences in angles at each interface due to refraction. Thus:

$$\delta = (\theta_1 - \theta_2) + (\theta_4 - \theta_3) = \theta_1 + \theta_4 - (\theta_2 + \theta_3)$$

Of course, one can prove this rigorously with quite a bit more geometry, but there is no need: we know physically what the deviation angle is, and can translate that to a nice mathematical formula. Now we can use the expression for φ in our last equation:

$$\delta = \theta_1 + \theta_4 - \varphi$$

We were given $\theta_1 = 50°$, so now we really just need to find θ_4 and we are done. From Snell's law above, we can relate θ_4 to θ_3 easily. We can also relate θ_3 to θ_2 and the apex angle of the prism, φ. Finally, we can relate θ_2 back to θ_1 with Snell's law. First, let us write down all the separate relations:

$$\begin{aligned}
\sin\theta_4 &= n\sin\theta_3 \\
\theta_3 &= \varphi - \theta_2 \\
n\sin\theta_2 &= \sin\theta_1 \\
\text{or}\quad \theta_2 &= \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)
\end{aligned}$$

If we put all these together (in the right order) we have θ_4 in terms of known quantities:

$$\begin{aligned}
\sin\theta_4 &= n\sin\theta_3 \\
&= n\sin\left(\varphi - \theta_2\right) \\
&= n\sin\left[\varphi - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right]
\end{aligned}$$

With that, we can write the full expression for the deviation angle:

$$\delta = \theta_1 + \theta_4 - \varphi = \theta_1 + n\sin\left[\varphi - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right] - \varphi$$

Now we just need to calculate the deviation separately for red and violet light, using their different indices of refraction. You should find:

$$\begin{aligned}
\delta_{\text{red}} &= 48.56° \\
\delta_{\text{blue}} &= 53.17°
\end{aligned}$$

The angular dispersion is just the difference between these two:

$$\text{angular dispersion} = \delta_{\text{blue}} - \delta_{\text{red}} = 4.62°$$

2. *Hecht 3.48* Show that for substances of low density, such as gases, which have a single resonant frequency ω_o, the index of refraction is given by

$$n \approx 1 + \frac{Nq_e^2}{2\epsilon_o m_e \left(\omega_o^2 - \omega^2\right)} \tag{1}$$

Solution: For an oscillator with $q = e$ of single resonance frequency without damping, we have (3.70) in Hecht:

$$n(\omega) = \sqrt{1 + \frac{Ne^2}{\epsilon_2 m_e} \frac{1}{\omega_o^2 - \omega^2}} \tag{2}$$

This comes from the relationships

$$n = \frac{\epsilon}{\epsilon_o} \tag{3}$$

$$\epsilon = \epsilon_o + \frac{P(t)}{E(t)} \qquad \text{(dielectric constant)} \tag{4}$$

$$P(t) = eNx(t) \qquad \text{(polarization)} \tag{5}$$

and $x(t)$ and $E(t)$ are the position and driving electric field of our oscillating charge derived previously (3.63, 3.66 in Hecht). At low density, we may suppose that the second term under the radical in Eq. 2 is small compared to 1. That is,

$$\frac{Ne^2}{2\epsilon_o m_e \left(\omega_o^2 - \omega^2\right)} \ll 1 \tag{6}$$

This is valid at low enough density N, provided the frequency ω is not right at the resonance frequency ω_o. Specifically, let us imagine that we are dealing with 200 nm UV light, $\omega \sim 1 \times 10^{16}$ Hz. If we consider driving frequencies in the visible, no lower than 350 nm ($\omega_o \sim 5 \times 10^{15}$ Hz), then $(\omega_o^2 - \omega^2)^{-1} \sim 10^{-32}$. Given that $e^2/\epsilon_o m_e \sim 3000$, our limit on density is

$$1 \gg \frac{Ne^2}{2\epsilon_o m_e \left(\omega_o^2 - \omega^2\right)} \tag{7}$$

$$N \ll \frac{2\epsilon_o m_e}{e^2} \left(\omega_o^2 - \omega^2\right) \sim 10^{35}\,\mathrm{m}^{-3} \tag{8}$$

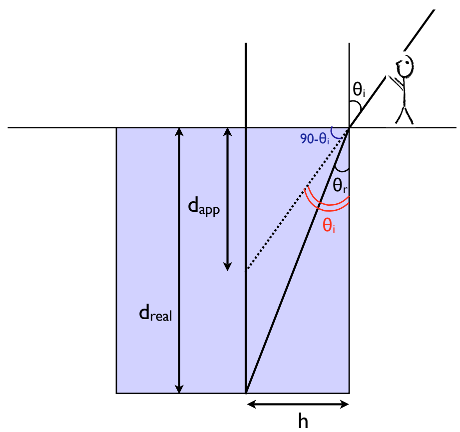
Our atmosphere at sea level has $\sim 10^{25}$ molecules per m^3, so this is a well-justified approximation for gases at visible frequencies, so long as we are reasonably far from the resonance frequency. Given the inequality in Eq. 6, we may approximate the radical in Eq. 2 with $\sqrt{1+x} \approx 1 + x/2$ since $x \ll 1$.

Thus,

$$n(\omega) = \sqrt{1 + \frac{Ne^2}{\epsilon_2 m_e} \frac{1}{\omega_o^2 - \omega^2}} \approx 1 + \frac{Ne^2}{2\epsilon_2 m_e} \frac{1}{\omega_o^2 - \omega^2} \tag{9}$$

3. What is the apparent depth of a swimming pool in which there is water of depth 3 m, **(a)** When viewed from normal incidence? **(b)** When viewed at an angle of 60° with respect to the surface? The refractive index of water is 1.33.

Solution: As always, we first need to draw a little picture of the situation at hand.



It is slightly more convenient to redefine the angle of incidence θ_i to be with respect to the *normal* of the water's surface itself, rather than with respect to the surface, since that is our usual convention. That means we are interested in incident angles for the observer of 90° and 30°. The depth of the pool will be $d_{\text{real}} = 3\,\text{m}$. If an observer views the bottom of the pool with an angle θ_i with respect to the surface normal, refracted rays from the bottom of the pool will be bent away from the surface normal on the way to their eyes. That is, rays emanating from the bottom of the pool will make an angle $\theta_r < \theta_i$ with respect to the surface normal, and rays exiting the pool will make an angle θ_i with the surface normal. This is owing to the fact that the light will be bent *toward* the normal in the faster medium, the air, on exiting the water.

What depth does the observer actually see? They see what light would do in the absence of refraction, the path that light rays would appear to take if the rays were not "bent" by the water. In this case, that means that the observer standing next to the pool would think they saw the light rays coming from an angle θ_i with respect to the surface normal (dotted line in the pool). The *lateral* position of the bottom of the pool would remain unchanged. If the real light rays intersect the bottom of the pool a distance h from the edge, then the apparent bottom of the pool is also a distance h from the edge of the pool. Try demonstrating this with a drinking straw in a glass of water!

So what to do? First off, we can apply Snell's law. If the index of refraction of air is 1, and the water has an index of refraction n, then

$$n \sin \theta_r = \sin \theta_i$$

We can also use the triangle defined by d_{real} and h:

$$\tan \theta_r = \frac{h}{d_{real}}$$

as well as the triangle defined by d_{real} and h[i]:

$$\tan (90 - \theta_i) = \frac{d_{app}}{h} = \frac{1}{\tan \theta_i}$$

Solving the last two equations for h,

$$\begin{aligned} h &= d_{real} \tan \theta_r = d_{app} \tan \theta_i \\ \implies \quad d_{app} &= d_{real} \left[\frac{\tan \theta_r}{\tan \theta_i} \right] \end{aligned}$$

From Snell's law, we have a relationship between θ_r and θ_i already:

$$\theta_r = \sin^{-1} \left[\frac{\sin \theta_i}{n} \right]$$

Putting everything together,

$$d_{app} = \frac{d_{real}}{\tan \theta_i} \tan \theta_r = \frac{d_{real}}{\tan \theta_i} \left[\tan \left(\sin^{-1} \left[\frac{\sin \theta_i}{n} \right] \right) \right]$$

If you just plug in the numbers at this point, you have a problem. One of the angles is $\theta_i = 0$, normal incidence, which means we have to divide by zero in the expression above. Dividing by zero is worse than drowning kittens, far worse. Thankfully, we know enough trigonometry to save the poor kittens.

We can save the kittens by remembering an identity for $\tan \left[\sin^{-1} x \right]$. If we have an equation like $y = \sin^{-1} x$, it implies $\sin y = x$. This means y is an angle whose sine is x. If y is an angle in a right

[i] Along with an identify for $\tan \theta$, *viz.*, $\tan (90 - \theta) = 1/ \tan \theta$

triangle, then it has an opposite side x and a hypotenuse 1, making the adjacent side $\sqrt{1-x^2}$. The tangent of angle y must then be $x/\sqrt{1-x^2}$. Thus,

$$\tan\left[\sin^{-1}x\right] = \frac{x}{\sqrt{1-x^2}}$$

Using this identity in our equation for d_{app},

$$d_{app} = \frac{d_{real}}{\tan\theta_i}\left[\frac{\sin\theta_i}{n\sqrt{1-\left[\frac{\sin\theta_i}{n}\right]^2}}\right] = \frac{d_{real}}{\tan\theta_i}\left[\frac{\sin\theta_i}{\sqrt{n^2-\sin^2\theta_i}}\right] = \frac{d_{real}\cos\theta_i}{\sqrt{n^2-\sin^2\theta_i}}$$

Viewed from normal incidence with respect to the surface means $\theta_i = 0$ – looking straight down at the surface of the water. In this case, $\sin\theta_i = 0$, and the result is simple:

$$d_{app} = \frac{d_{real}}{n} \approx 2.25\,\mathrm{m}$$

Viewed from 60° with respect to the *surface* means 30° with respect to the normal, and thus

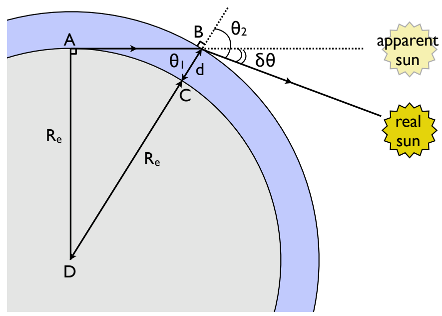
$$d_{app} = d_{real}\cos 30\left[\frac{1}{\sqrt{1.33^2-\sin^2 30}}\right] \approx 2.1\,\mathrm{m}$$

There are easier ways to solve the normal incidence problem, without endangering any kittens whatsoever. Solving that problem, however, is a special case, and of limited utility. You would still have to solve the case of 60° incidence separately. I wanted to show you here that solving the general problem just once is all you need to do, so long as you are careful enough.

4. As light from the Sun enters the atmosphere, it refracts due to the small difference between the speeds of light in air and in vacuum. The optical length of the day is defined as the time interval between the instant when the top of the Sun is just visibly observed above the horizon, to the instant at which the top of the Sun just disappears below the horizon. The geometric length of the day is defined as the time interval between the instant when a geometric straight line drawn from the observer to the top of the Sun just clears the horizon, to the instant at which this line just dips below the horizon. The day's optical length is slightly larger than its geometric length.

By how much does the duration of an optical day exceed that of a geometric day? Model the Earth's atmosphere as uniform, with index of refraction $n = 1.000293$, a sharply defined upper surface, and depth 8767 m. Assume that the observer is at the Earth's equator so that the apparent path of the rising and setting Sun is perpendicular to the horizon. You may take the radius of the earth to be 6.378×10^6 m. Express your answer to the nearest hundredth of a second.

Solution: First, we need to draw a little picture. This is the situation we have been given:



We presume that some human is standing at point A on the earth's surface, looking straight out toward the horizon. This line of sight intersects the boundary between the atmosphere and space (which we are told to assume is a sharp one) at point B. Light rays from the sun, which is slightly below the horizon, are refracted toward the earth's surface at point B, and continue on along the line of sight from B to A. We know the index of refraction of vacuum is just unity ($n_{\text{vacuum}} = 1$), while that of the atmosphere is $n = 1.000293$. The day appears to be slightly longer because we see the sun even after it has gone through an extra angle of rotation $\delta\theta$ due to atmospheric refraction.

To set up the geometry, we first draw a radial line from point B to the center of the earth. This line, $\overline{BC}$, will intersect the boundary of the atmosphere at point B, and will be normal to the atmospheric boundary. This defines the angle of incidence θ_2 and the angle of refraction θ_1 for light coming from the sun. The difference between these two angles, $\delta\theta$, is how much the light is bent downward upon being refracted from the atmosphere. How do we relate this to the extra length of the day one would observe? We know that the earth revolves on its axis at a constant angular speed - one revolution in 24 hours. Thus, we can easily find the angular speed of the earth:

$$\text{earth's angular speed} = \omega = \frac{\text{one revolution}}{1\text{day}} = \frac{360°}{86400\,\text{s}}$$

Here we used the fact that there are $24 \cdot 60 \cdot 60 = 86400$ seconds in one day. Given the angular velocity of the earth, we know exactly how long it will take for the earth to rotate through the "extra" angle $\delta\theta$ due to refraction:

$$\delta\theta = \omega\delta t$$

We only need one last bit: the atmospheric refraction occurs *twice per day* – once at sun-up and once at sun-down. The total "extra" length of the day is then $2\delta t$. Thus, if we can find $\delta\theta$, we can figure out how much longer the day seems to be due to atmospheric refraction. In order to find it, we need to use the law of refraction and a bit of geometry. First, from the law of refraction and the fact that $\delta\theta = \theta_2 - \theta_1$, we can state the following:

$$\theta_2 - \theta_1 = \delta\theta$$
$$n \sin\theta_1 = \sin\theta_2 = \sin(\theta_1 + \delta\theta)$$

In order to proceed further, we draw a line from point A to the center of the earth, point D. This forms a triangle, $\triangle ABD$. Because line $\overline{AD}$ is a radius of the earth, by construction, it must intersect line $\overline{AB}$ at a right angle, since the latter is by construction a tangent to the earth's surface. Thus, $\triangle ABD$ is a right triangle, and

$$\sin\theta_1 = \frac{\overline{AD}}{\overline{BD}} = \frac{R_e}{R_e + d}$$

Plugging this into the previous equation,

$$n \sin\theta_1 = \sin\theta_2 = \sin(\theta_1 + \delta\theta) = n\frac{R_e}{R_e + d}$$

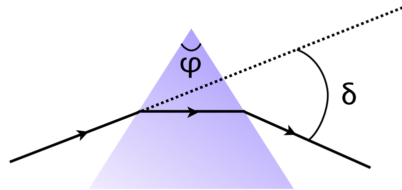
In principle, we are done at this point. The previous expression allows one to calculate θ_1, while the present one allows one to find $\delta\theta$ if θ_1 is known. From that, one only needs the angular speed of the earth.

$$\theta_2 = \theta_1 + \delta\theta = \sin^{-1}\left[\frac{nR_e}{R_e + d}\right]$$
$$\delta\theta = \sin^{-1}\left[\frac{nR_e}{R_e + d}\right] - \theta_1 = \sin^{-1}\left[\frac{nR_e}{R_e + d}\right] - \sin^{-1}\left[\frac{R_e}{R_e + d}\right] = \omega\delta t$$
$$2\delta t = \frac{2\delta\theta}{\omega} \approx 163.82\,\text{s}$$

5. Light is deviated by a glass prism of index n as shown in the figure below. The ray in the prism is parallel to the base. Show that the refractive index is related to the deviation angle δ and the prism angle φ by the equation

$$n \sin\frac{\varphi}{2} = \sin\left(\frac{\varphi + \delta}{2}\right)$$

for this angle of incidence (*i.e.,*, the angle of incidence such that the ray in the prism is parallel to the base). The deviation angle δ is a minimum for this angle of incidence, and is known as the angle of minimum deviation. *Hint: You can solve this and the first problem together, if you keep things as general as possible - this is just a special case of the first problem.*



Solution: Refer to the solution to the first problem. We know already how to relate the deviation angle to the incident, exit, and refracted angles:

$$\delta = \theta_1 + \theta_4 - (\theta_2 + \theta_3) \tag{10}$$

We also know how to relate the incident, exit, and refracted angles to the prism angle:

$$\varphi = \theta_2 + \theta_3 \tag{11}$$

Since the ray in the prism is parallel to the base, we know that $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$ by symmetry. This gives us

$$\varphi = 2\theta_2 \tag{12}$$

$$\delta + \varphi = 2\theta_1 \tag{13}$$

Dividing the last equation by two and taking the sin of both sides,

$$\sin\left(\frac{\delta + \varphi}{2}\right) = \sin\theta_1 \tag{14}$$

Snell's law lets us relate θ_1 to θ_2, and we know $\theta_2 = \varphi/2$.[ii] Thus,

$$\sin\left(\frac{\delta + \varphi}{2}\right) = \sin\theta_1 = n\sin\theta_2 = n\sin\frac{\varphi}{2} \tag{15}$$

6. *Hecht 3.57* In 1871 Sellmeier derived the equation

$$n^2 = 1 + \sum_j \frac{A_j \lambda^2}{\lambda^2 - \lambda_{0j}^2} \tag{16}$$

where the A_j terms are constants and each λ_{0j} is the vacuum wavelength associated with a natural

[ii] We could also use the fact that $\theta_2 = \theta_3 = 90 - \left(90 - \frac{\varphi}{2}\right) = \varphi/2$ – from the figure in the solution to problem 1, $\angle ACB$ is $90 - \varphi/2$, and $\angle ACB$ plus θ_3 make up a right angle.

frequency ν_{0j}, such that $\lambda_{0j}\nu_{0j} = c$. This formulation is a considerable practical improvement over the Cauchy equation. Show that where $\lambda \gg \lambda_{0j}$, Cauchy's equation is an approximation of Sellmeier's. *Hint:* write the above expression with only the first term in the sum; expand it by the binomial theorem; take the square root of n^2 and expand again.

Solution: The Cauchy equation (problem 3.54) is

$$n = C_1 + \frac{C_2}{\lambda^2} + \frac{C_3}{\lambda^4} + \dots \tag{17}$$

The Sellmeier equation can be rewritten

$$n^2 = 1 + \sum_j \frac{A_j}{1 - \lambda_{0j}^2/\lambda^2} \tag{18}$$

if we presume $\lambda_{0j}/\lambda \ll 1$, we may use the approximation $1/(1-x) \approx 1 + x$:

$$n^2 \approx 1 + \sum_j A_j \left(1 + \lambda_{0j}^2/\lambda^2\right) \tag{19}$$

Taking the square root and rearranging,

$$n \approx \sqrt{1 + \sum_j A_j \left(1 + \lambda_{0j}^2/\lambda^2\right)} = \sqrt{\left(1 + \sum_j A_j\right) + \sum_j A_j \lambda_{0j}^2/\lambda^2} \tag{20}$$

$$\approx \sqrt{1 + \sum_j A_j}\sqrt{1 + \frac{\sum_j A_j \lambda_{0j}^2/\lambda^2}{1 + \sum_j A_j}} \tag{21}$$

The first term is simply a constant. In the second term, given that $\lambda_{0j}/\lambda \ll 1$, the sum in the numerator will always be much less than $\sum_j A_j$ (which is just a constant), ensuring that the fraction is always much less than 1. Thus, the approximation $\sqrt{1+x} \approx 1 + x/2$ is warranted.

$$n \approx \sqrt{1 + \sum_j A_j}\left(1 + \frac{\sum_j A_j \lambda_{0j}^2/\lambda^2}{2\left(1 + \sum_j A_j\right)}\right) = \sqrt{1 + \sum_j A_j} + \frac{\sum_j A_j \lambda_{0j}^2/\lambda^2}{2\sqrt{1 + \sum_j A_j}} = C_1 + \frac{C_2}{\lambda^2} \tag{22}$$

This is equivalent to Cauchy's equation up to second order in $1/\lambda$ (we should not expect higher-order agreement, given the approximations above), with

$$C_1 \approx \sqrt{1 + \sum_j A_j} \tag{23}$$

$$C_2 \approx \frac{\sum_j A_j \lambda_{0j}^2}{2\sqrt{1 + \sum_j A_j}} \tag{24}$$

Had we carried our approximations to higher order, we could have found the C_3 and C_4 coefficients. If we consider only a single term ($j=1$), things are much simpler:

$$C_1 \approx \sqrt{1 + A} \tag{25}$$

$$C_2 \approx \frac{A\lambda_0^2}{2\sqrt{1 + A}} \tag{26}$$

One could consider only a single term in the sums much earlier and make things a little less messy, but that seems lame.

7. *Hecht 4.5* Imagine that we have a non-absorbing glass plate of index n and thickness Δy, which stands between a source S and observer P.

(a) If the unobstructed wave (without the plate present) is $E_u = E_o \exp i\omega\,(t - y/c)$, show that with the plate in pace the observer sees a wave

$$E_p = E_o \exp i\omega\,[t - (n-1)\,\Delta y/c - y/c] \tag{27}$$

(b) Show that if either $n \approx 1$ or Δy is very small, then

$$E_p = E_u + \frac{\omega\,(n-1)\,\Delta y}{c} E_u e^{-i\pi/2} \tag{28}$$

The second term on the right may be envisioned as the field arising from the oscillators in the plate.

Solution: **(a)** Without any media present, to cross a distance Δy at speed c the wave will take a time

$$t_{wo} = \frac{\Delta y}{c} \tag{29}$$

With the medium present, the wave has a reduced speed of $v = c/n$, and the time required is

$$t_w = \frac{n\Delta y}{c} \tag{30}$$

Thus, the presence of the media delays the wave by an amount

$$\delta t = t_w - t_{wo} = \frac{\Delta y}{c}(n-1) \tag{31}$$

This time delay shows up as an extra phase compared to the original unimpeded wave. That is, since the wave in the presence of the glass is delayed by a time δt compared to the unimpeded wave, we must shift the time coordinate back by that amount. This amounts to making the substitution $t \to t - \delta t$ in the original wave to account for the time lag, or equivalently, multiplying by a phase factor $e^{-i\omega\delta t}$

$$E_p = E_u(t-\delta t) = E_o e^{i\omega(t-\delta t - y/c)} = E_o e^{i\omega(t-(n-1)\Delta y/c - y/c)} \tag{32}$$

$$= E_o e^{i\omega(t-y/c)} e^{-i\omega(n-1)\Delta y/c} = E_u e^{-i\omega\delta t} \tag{33}$$

The latter form makes it clear that the magnitude of E is unchanged by the presence of the glass, only the phase is altered.

(b) If $(n-1)\Delta y \ll 1$, we may use the approximation $e^x \approx 1+x$ for the phase factor:

$$e^{-i\omega\delta t} = e^{-i\omega(n-1)\Delta y/c} \approx 1 - i\omega(n-1)\Delta y/c \tag{34}$$

Noting that $-i = e^{-i\pi/2}$,

$$e^{-i\omega\delta t} \approx 1 + \omega(n-1)\frac{\Delta y}{c}e^{-i\pi/2} \tag{35}$$

Substituting back into the equation for E_p,

$$E_p = E_u e^{-i\omega\delta t} = E_u\left(1 + \omega(n-1)\frac{\Delta y}{c}e^{-i\pi/2}\right) = E_u + \frac{\omega(n-1)\Delta y}{c}E_u e^{-i\pi/2} \tag{36}$$

The observer sees a wave which is the sum of two terms: the first is the wave that would be observed without the glass present, the second is a $\pi/2$ phase-shifted interference term, representing the out-of-phase field due to the oscillators in the glass plate.

8. *Hecht 4.29* Starting with Snell's law, prove that the vector refraction equation has the form

$$n_t \hat{k}_t - n_i \hat{k} = (n_t \cos\theta_t - n_i \cos\theta_i)\, \hat{u}_n \tag{37}$$

Solution: (Refer to figures 4.38 and 4.39 in Hecht.) The unit vectors $\hat{k}_i$ and $\hat{k}_t$ point along the directions of the incident and transmitted (refracted) waves, respectively, while $\hat{u}_n$ points along the interface normal, from the incident to transmitted media. Snell's law reads

$$n_i \sin\theta_i = n_t \sin\theta_t \tag{38}$$

where n_i and n_t are the indices of refraction in the incident and transmitted media, and θ_i and θ_t are the angles that the incident and transmitted waves make with respect to the interface normal. We can rewrite each side of Snell's law as a vector equation

$$n_i \sin\theta_i = \left| n_i \left(\hat{k}_i \times \hat{u}_n \right) \right| \tag{39}$$

$$n_t \sin\theta_t = \left| n_i \left(\hat{k}_t \times \hat{u}_n \right) \right| \tag{40}$$

Since $\hat{k}_i$, $\hat{k}_t$, and $\hat{u}_n$ lie within the same plane, $\hat{k}_i \times \hat{u}_n$ and $\hat{k}_t \times \hat{u}_n$ point along the same direction perpendicular to that plane. Thus, we may combine the above to generalize Snell's law to

$$n_i \left(\hat{k}_i \times \hat{u}_n \right) = n_i \left(\hat{k}_t \times \hat{u}_n \right) \tag{41}$$

or

$$n_t \left(\hat{k}_t \times \hat{u}_n \right) - n_i \left(\hat{k}_t \times \hat{u}_n \right) = \left(n_t \hat{k}_t - n_i \hat{k}_i \right) \times \hat{u}_n = 0 \tag{42}$$

This means that the vector $\vec{\Gamma} = n_t \hat{k}_t - n_i \hat{k}_i$ must be parallel to $\hat{u}_n$, since it has a cross product of zero with $\hat{u}_n$. We can thus write it in terms of a scalar magnitude times $\hat{u}_n$, $\vec{\Gamma} = \Gamma \hat{u}$, where Γ is known as the *astigmatic constant*. It is the difference between the projections of $n_t \hat{k}_t$ and $n_i \hat{k}_i$ on $\hat{u}_n$, which we can make apparent if we take the dot product of Γ with $\hat{u}_n$:

$$\left(n_t \hat{k}_t - n_i \hat{k}_i \right) \cdot \hat{u}_n = n_t \left(\hat{k}_t \cdot \hat{u}_n \right) - n_i \left(\hat{k}_i \cdot \hat{u}_n \right) \tag{43}$$

The dot product $\hat{k}_i \cdot \hat{u}_n$ is the projection of $\hat{k}_i$ along the direction perpendicular to the interface, which is just the cosine of the angle of incidence: $\hat{k}_i \cdot \hat{u}_n = \cos\theta_i$. A similar relationship holds for the transmitted wave:

$$\vec{\Gamma} \cdot \hat{u}_n = \left(n_t \hat{k}_t - n_i \hat{k}_i\right) \cdot \hat{u}_n = n_t \cos\theta_t - n_i \cos\theta_i \tag{44}$$

The scalar on the right side of the equation above is just the magnitude of $\vec{\Gamma}$, given that $\vec{\Gamma}$ points along $\hat{u}_n$, so we may write

$$\vec{\Gamma} = n_t \hat{k}_t - n_i \hat{k}_i = (n_t \cos\theta_t - n_i \cos\theta_i)\,\hat{u}_n \tag{45}$$

For completeness, the astigmatic constant must be $\Gamma = n_t \cos\theta_t - n_i \cos\theta_i$.

Alternate Solution 1 (Joseph Murray):

Given Snell's law

$$n_i \sin\theta_i = n_t \sin\theta_t \tag{46}$$

we can translate into vector form easily

$$n_i \left(\hat{k}_i \times \hat{u}_n\right) = n_t \left(\hat{k}_t \times \hat{u}_n\right) \tag{47}$$

Cross $\hat{u}_n$ into both sides:

$$n_I \left(\hat{u}_n \times \hat{k}_i \times \hat{u}_n\right) = n_t \left(\hat{u}_n \times \hat{k}_t \times \hat{u}_n\right) \tag{48}$$

Now note that

$$\vec{a} \times \vec{b} \times \vec{c} = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\left(\vec{a} \cdot \vec{b}\right) \tag{49}$$

and thus

$$n_i \left[\hat{k}_i\,(\hat{u}_n \cdot \hat{u}_n) - \hat{u}_n \left(\hat{u}_n \cdot \hat{k}_i\right)\right] = n_t \left[\hat{k}_t\,(\hat{u}_n \cdot \hat{u}_n) - \hat{u}_n \left(\hat{u}_n \cdot \hat{k}_t\right)\right] \tag{50}$$

$$\implies \quad n_i \hat{k}_i - n_i \cos(\theta_i) \hat{u}_n = n_t \hat{k}_t - n_t \cos(\theta_t) \hat{u}_n \tag{51}$$

$$\implies \quad n_i \hat{k}_i - n_t \hat{k}_t = (n_i \cos\theta_i - n_t \cos\theta_t)\,\hat{u}_n \tag{52}$$

Alternate Solution 2 (Andrew Tuggle):

We start with Snell's law as a cross-product:

$$n_i\left(\hat{k}_i \times \hat{u}_n\right) = n_t\left(\hat{k}_t \times \hat{u}_n\right) \tag{53}$$

and distribute $\hat{u}_n$:

$$\left(n_t\hat{k}_t - n_i\hat{k}_i\right) = 0 \tag{54}$$

to find that $n_t\hat{k}_t - n_i\hat{k}_i$ is parallel to $\hat{u}_n$, i.e.,

$$n_t\hat{k}_t - n_i\hat{k}_i = A\hat{u}_n \tag{55}$$

Squaring both sides, and noting $n_t \sin\theta_t = n_i \sin\theta_i$,

$$A^2 = n_t^2 + n_i^2 - 2n_t n_i \cos(\theta_t - \theta_i) \tag{56}$$

$$= n_t^2 + n_i^2 - 2n_t n_i (\cos\theta_i \cos\theta_t + \sin\theta_i \sin\theta_t) \tag{57}$$

$$= (n_t \cos\theta_t - n_i \cos\theta_i)^2 + (n_t \sin\theta_t - n_i \sin\theta_i)^2 \tag{58}$$

$$= (n_t \cos\theta_t - n_i \cos\theta_i)^2 \tag{59}$$

Then,

$$A = n_t \cos\theta_t - n_i \cos\theta_i \tag{60}$$

and thus

$$n_t\hat{k}_t - n_i\hat{k}_i = (n_t \cos\theta_t - n_i \cos\theta_i)\,\hat{u}_n \tag{61}$$

Alternate Solution 2 (Joseph Lukens):

Follow the conventions on pg. 103 of the textbook. Snell's law states

$$n_i \sin\theta_i = n_t \sin\theta_t \tag{62}$$

If we resolve the unit propagation vectors $\hat{k}_i$ and $\hat{k}_t$ into their components normal and tangential to the surface, we find

$$\hat{k}_i = \cos\theta_i \hat{u}_n + \sin\theta_i \hat{u}_t \quad \Longrightarrow \quad \sin\theta_i \hat{u}_t = \hat{k}_i - \cos\theta_i \hat{u}_n \tag{63}$$

$$\hat{k}_t = \cos\theta_t \hat{u}_n + \sin\theta_t \hat{u}_t \quad \Longrightarrow \quad \sin\theta_t \hat{u}_t = \hat{k}_t - \cos\theta_t \hat{u}_n \tag{64}$$

Snell's law also implies that

$$n_i \sin\theta_i \hat{u}_t = n_t \sin\theta_t \hat{u}_t \tag{65}$$

so that we can plug in the expressions for $\sin\theta_i \hat{u}_t$ and $\sin\theta_t \hat{u}_t$ obtained earlier to yield

$$n_i \left(\hat{k}_i - \cos\theta_i \hat{u}_n\right) = n_t \left(\hat{k}_t - \cos\theta_t \hat{u}_n\right) \Longrightarrow \quad n_t \hat{k}_t - n_i \hat{k}_i = \left(n_t \cos\theta_t - n_i \cos\theta_i\right) \hat{u}_n \tag{66}$$