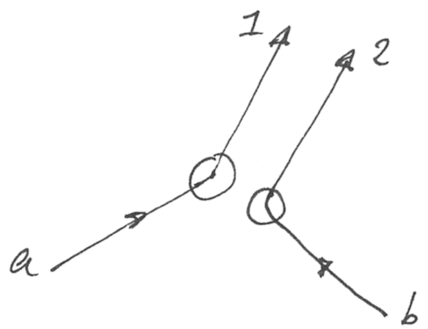
Addition rules for Bose particles have interesting consequences

- leads to Blackbody directly
- LASERS!

Go back to scattering. 2 ~~Bose~~ different Bose particles & 2 $\boxed{\text{different}}$ scatterers

details not key; what happens to scattered particles?




a scattered into state 1

b scattered into state 2

- state = given direction & energy, etc
- presume states 1 & 2 are basically same (isotropic space, e.g.)

<u>What if we only had particle a</u>? Then we know

$$\text{amplitude } a \to 1 : \quad \langle 1|a\rangle$$

and for b alone, $\langle 2|b\rangle$

- if a and b $\boxed{\text{were not identical particles}}$, the amplitude for both scatterings occuring simultaneously is

$$\langle 1|a\rangle\langle 2|b\rangle \qquad a \neq b$$

so probability is $\quad P = |\langle 1|a\rangle\langle 2|b\rangle|^2 = \underbrace{|\langle 1|a\rangle|^2}_{a\to 1}\underbrace{|\langle 2|b\rangle|^2}_{b\to 2}$

independent, distinguishable, so indiv prob mult.

$$P = P_{1a} \cdot P_{2b} \qquad \text{(like rolling dice twice)}$$

Shorthand : $P_{1a} = |\langle 1|a\rangle|^2 = |a_1|^2$ or $\langle 1|a\rangle = a_1$ ampl ~~...~~ $a \to 1$ alone

$P_{2b} = |\langle 2|b\rangle|^2 = |b_2|^2$ or $\langle 2|b\rangle = b_2$

So for double scattering, $P = |a_1|^2 |b_2|^2$

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Now: if particle ~~[illegible]~~ $a$ & $b$ are similar, but still not identical, it is equally likely that $b \to 1$ and $a \to 2$! ~~can't tell apart~~

ampl $\langle 2|a\rangle\langle 1|b\rangle$

$P = |\langle 2|a\rangle\langle 1|b\rangle|^2 = |a_2|^2 |b_1|^2$ 2 outcomes; disting, equally likely

run this scattering many times ... probability of getting 2 particles at posns 1 and 2 simultaneously = sum of prob. of 2 ways to do it!

$P_2 = |a_1|^2 |b_2|^2 + |a_2|^2 |b_1|^2$ 2 distinguishable events, either or ~~indistinguishable events~~

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~~• now, since we have Bose particles, both could end up in the same state!~~

~~i.e. $1 = 2$ then $a_1 = a_2 = a$ $b_1 = b_2 = b$~~

~~then $P_2 = 2|a|^2 |b|^2$ prob. of particles a and b both scattering into same state~~

Suppose we let the directions 1 and 2 get very close together

then $a_1 \approx a_2$ $\quad b_1 \approx b_2$

or just $a_1 = a_2 = a$ $\quad b_1 = b_2 = b$

then $P_2 = 2|a|^2|b|^2$ $\quad$ 2 distinct particles both end up @ same posn

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But: what if we have $\boxed{\text{Bose particles}}$ (identical)? Identical

then $a \to 1$ and $b \to 2$ CANNOT be distinguished

from $a \to 2$ and $b \to 1$

if we have 2 indistinguishable outcomes, we need to add amplitudes

total ampl $= \langle 1|a\rangle\langle 2|b\rangle + \langle 2|a\rangle\langle 1|b\rangle = a_1 b_2 + a_2 b_1$

So, if $1 = 2$ and $a = b$

$$P_2 = |a_1 b_2 + a_2 b_1|^2 = 4|a|^2|b|^2$$ (constructive interference!)

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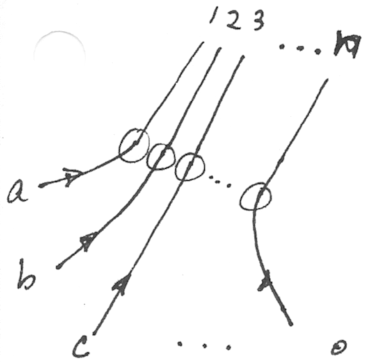
! it is twice as likely to find 2 identical Bose particles scattered into the same state

– compared – to assuming the particles were different!

$\Rightarrow$ if there is already 1 Bose particle in a given state, the amplitude of putting an identical one on top of it is $\sqrt{2}$ greater than if it weren't there

Suppose we have N bose particles?




- n particles; a, b, c ... <u>distinguishable</u> for now
- scattered into directions 1, 2, 3 ... n
- all directions lead into a small detector $\underset{\text{area}}{\Delta S}$

• probability that n particles will be counted together in n surface elements of detector

$$\underbrace{|a_1 b_2 c_3 \cdots|^2}_{\substack{\text{ampl} \\ a \to 1}} \cdot \underbrace{dS_1}_{\substack{\text{surface element} \\ \text{of detector}}} dS_2\, dS_3 \cdots$$

presume all dS across detector are the same

$\Rightarrow \quad |a|^2 |b|^2 |c^2| \cdots dS_1\, dS_2\, dS_3$

same ampl for <u>a to go to any $dS_i$</u>


integrate all dS over the surface of the detector $\Delta S$
(all $dS_i$ same)

$\Rightarrow \quad P_n$ (n <u>different</u> particles @ once) $= |a|^2 |b|^2 |c|^2 \cdots (\Delta S)^n$

- just product of the probabilities for each particle to enter separately

$$P_{tot} = P_1 P_2 P_3 \cdots P_n$$

• Now, suppose they are all identical Bose particles
now there are <u>many</u> indistinguishable possibilities

Say we have only 3 particles, then we have

$$\left\{\begin{array}{l} a\to 1\\ b\to 2\\ c\to 3\end{array}\right. \qquad \begin{array}{l} a\to 1\\ b\to 3\\ c\to 2\end{array} \qquad \begin{array}{l} a\to 2\\ b\to 1\\ c\to 3\end{array}$$

$$\begin{array}{l} a\to 2\\ b\to 3\\ c\to 1\end{array} \qquad \begin{array}{l} a\to 3\\ b\to 1\\ c\to 2\end{array} \qquad \begin{array}{l} a\to 3\\ b\to 2\\ c\to 1\end{array}$$

6 different combos $= 3!$

for $n$ particles: $n!$ different but indistinguishable chances

need to add amplitudes first

$$\boxed{P(n \text{ particles in } n \text{ surf elements}) = |\underbrace{a_1 b_2 c_3 + \ldots + a_1 b_3 c_2 \ldots}_{\substack{\text{all combos} \\ n! \text{ of them}}}| dS_1 dS_2 \ldots}$$

if the detector is small, and directions are very close together,

then $a_1 = a_2 = \ldots a = a_n$ & so on

i.e., all outcomes have same chance

$$\Rightarrow \boxed{P(n\,part) = |\underbrace{n!}_{\#\text{ combos}}\,\underbrace{abc}_{\text{any one combo}}|^2 dS_1 dS_2 \cdots dS_n}$$

if we integrate over the whole detector, each possible product of surface elements is counted $n!$ times ... so divide by $n!$ $\int dS \to \Delta S$

$$\boxed{P_n(Bose) = \frac{1}{n!}|n!\,abc\ldots|^2(\Delta S)^n = n!\,|abc\ldots|^2 \Delta S^n}$$

! probability of counting $n$ Bose particles together is $n!$ greater than if they were distinguishable! $\boxed{P_n(Bose) = n!\,P_n(diff)}$

So? what is the probability that a Bose particle goes into a particular state when there are already $n$ others present?

say the newly-added particle is $\boxed{w}$. With $(n+1)$ particles already present

$$P_{n+1}(\text{Bose}) = (n+1)!\,|abc\cdots w|^2(\Delta S)^{n+1}$$

$$= (n+1)\,\underbrace{|w|^2\Delta S}\;\underbrace{P_n(\text{Bose})}$$

prob of getting $n$ particles

probability of getting $w$ into detector if no other particles were present

for Bose particles, probability is enhanced by the $n$ already there!

- Presence of other particles increases the probability of getting one more!

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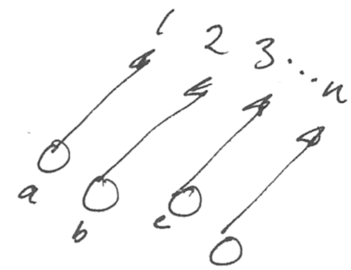
So? photons are Bose particles!

$n$ atoms $a, b, c \ldots$ emitting light




$\Rightarrow$ probability an atom will emit a photon into a particular final state is increased by a factor $(n+1)$ if there are already $n$ photons in that state

or $\boxed{\underbrace{\langle n+1|n\rangle}_{\text{adding }(n+1)^{st}\text{ photon to same state}} = \sqrt{n+1}\,a}$

$a = \langle i|a\rangle$

amplitude of photon into state $i$, no others present can reverse this too: ~~absorption~~

absorption is just decreasing the number!

$\text{ampl} = \langle n-1 | n \rangle = \sqrt{n}\, a^*$ also enhanced!

Say we have a box full of photons (like blackbody)

- n identical photons
- one atom in the box which can emit a photon into the same state

probability of emission is $(n+1)|a|^2 \Longrightarrow \boxed{P = (n+1) I(\omega)}$

absorbtion $n|a|^2$

prob of $(n+1)^{st}$ emission depends on intensity

- if $|a|^2$ is the emission probability with no other photons i.e., the INTENSITY as well!

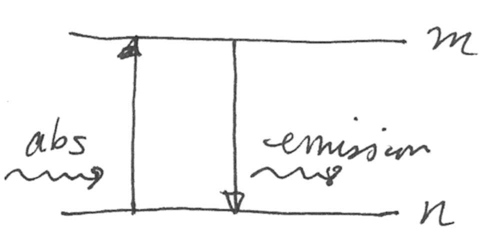
i.e., the presence of photons stimulates the emission/absorption of others!

## Einsteins laws of radiation

- now lets think again in terms of energy levels
- rate of transition from 1 level to another w/photon emission

Einstein proposed that when an atom has light of the right E shining on it, it can make a transition btw state

e.g. 2 level system




we know this...

- if photon energy matches $n \leftrightarrow m$ spacing, transition induced
- probability depends on the 2 levels (sel rules, $\Delta E$, etc)
- but it also depends on how intense the light is! ~~more int more chance to~~

rate of transitions from $m \to n$ = rate of photon emiss

$\boxed{\text{How to calculate?}}$

- even w/o other light, some chance an excited atom emits
  = spontaneous emission : $(\text{atom})^* \to (\text{atom}) + \text{photon}$
  - if an atom is excited, probability $A_{mn}$ for emission (jump $m \to n$) independent of other light
- as we just saw, if there are other photons of the same E there is stimulated emission! $m \to n$ enhanced by other photons
  (let innuendo sink in !!)

  prob. $B_{mn}$ $\boxed{(\text{atom}) + \text{photon} \to \text{atom} + 2\,\text{photons}}$
  chain Rx ?

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- 3 processes: absorption, spontaneous emission (indep int)
  (clearly depends on intensity)
  stimulated emission = (depends on intensity)

in equil @ T, number of atoms $N_n$ in state n
$N_m$ in state m

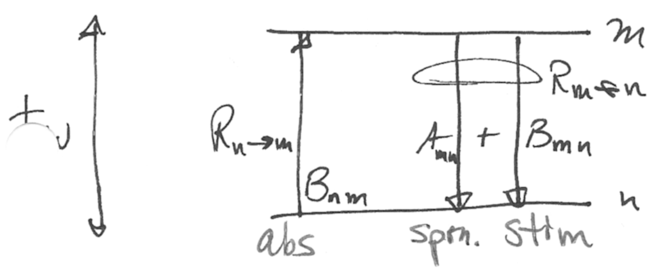
total # from $n \to m$? absorption depends on incident intensity
– number in $n$ times prob. of going to $m$

$\boxed{\text{rate:}}$ $R_{n\to m} = N_n \, B_{nm} \, I(\omega)$ (absorption is only "stimulated")

(# in $n$) (prob $\to m$) (intensity)

from $m \to n$ (emit) same way, but with stim & spont emission

$$R_{m\to n} = N_m \left[ \underset{\text{spont.}}{A_{mn}} + \underset{\text{stimulated}}{B_{mn}} I(\omega) \right]$$




one way up; 2 ways down

- in equilibrium, rate ↑ equals rate ↓ ... conservation
- we also know relative occupation in equil!

Boltzmann factor (like in Blackbody)

$$\frac{N_m}{N_n} = \frac{e^{-E_m/kT}}{e^{-E_n/kT}} = e^{-(E_m - E_n)/kT} = e^{-\hbar\omega/k_B T}$$

spacing of levels

since all photons have $E = E_m - E_n = \hbar\omega$

equating rates: $N_n B_{nm} I(\omega) = N_m [A_{mn} + B_{mn} I(\omega)]$

$$\Rightarrow \frac{N_n}{N_m} B_{nm} I(\omega) = B_{nm} I(\omega) e^{\hbar\omega/kT} = A_{mn} + B_{mn} I(\omega)$$

now we can get $I(\omega)$ in terms of these rates

$$\boxed{I(\omega) = \frac{A_{mn}}{B_{nm} e^{\hbar\omega/kT} - B_{mn}}}$$ like Planck ...

- last bit: $B_{nm} = B_{mn}$ to agree with Plank!

  or, absorption probability = stimulated emission prob!
  (remember: Box with photons!)

- further, to get Planck we require $\boxed{\frac{A_{mn}}{B_{mn}} = \frac{\hbar\omega^3}{\pi^2 c^2}}$

$\Rightarrow$ know absorption rate for a given level (easy to measure)

can deduce spontaneous & stimulated emission rates!

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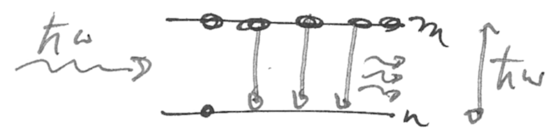
<u>Now we have the basis for a laser!</u>

- if light is present, it tends to induce the downward transition
- transition <u>adds</u> $\hbar\omega$ worth of light
- if we can make $N_m > N_n$, we can have huge light output!

- arrange by some means for most atoms to be in state $m$
  while few are in lower state $n$

(e.g, intense white light )
electric excitation




- light of frequency $\boxed{\omega = \frac{E_m - E_n}{\hbar}}$ will not be strongly absorbed, since most atoms are already excited

- on the other hand, if light of $\omega = \frac{E_m - E_n}{\hbar}$ is present, it will induce $m \longrightarrow n$ transition ~~of other~~

  ... which makes more $\omega$ photons

  ... which means more downward transitions ...

– Cascade / positive feedback! Chain Rx

– tiny light in at $\omega \Rightarrow$ huge light out at $\omega$!

$\Rightarrow$ light ampl by stimulated emission of rad = LASER

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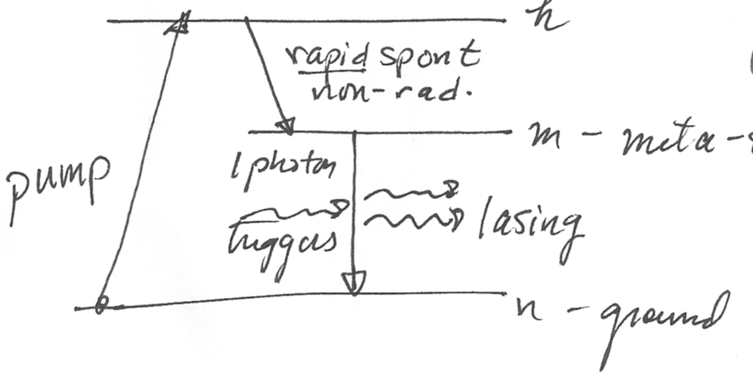
how to actually do this?

- we really need 3 levels, or all our excited states would fall back at the same rate — can't make $N_m > N_n$

further: different lifetimes




1 way up, 2 down

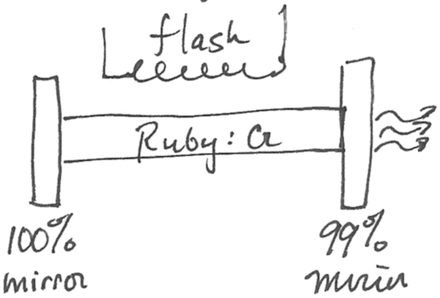
$\boxed{\text{one less likely ... bottleneck}}$

- using high-freq light or electric discharge excite most atoms to high state h

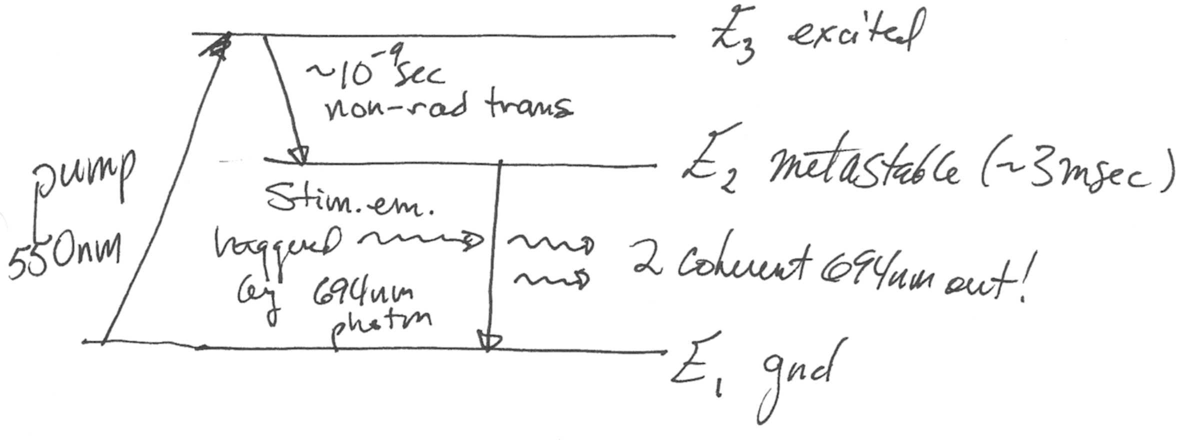
- quickly relax to m via non-radiative means (e.g. collisions)

- m is long-lived, over-populated (e.g. play w/ spin like para-ortho He) single matching $(E_m - E_n)$ photon triggers cascade! stimulated emiss.

e.g. Ruby: $Cr^{3+}$ ~0.1%




- many reflections ⇒ more chance to stimulate emission
- flash light excites crystal

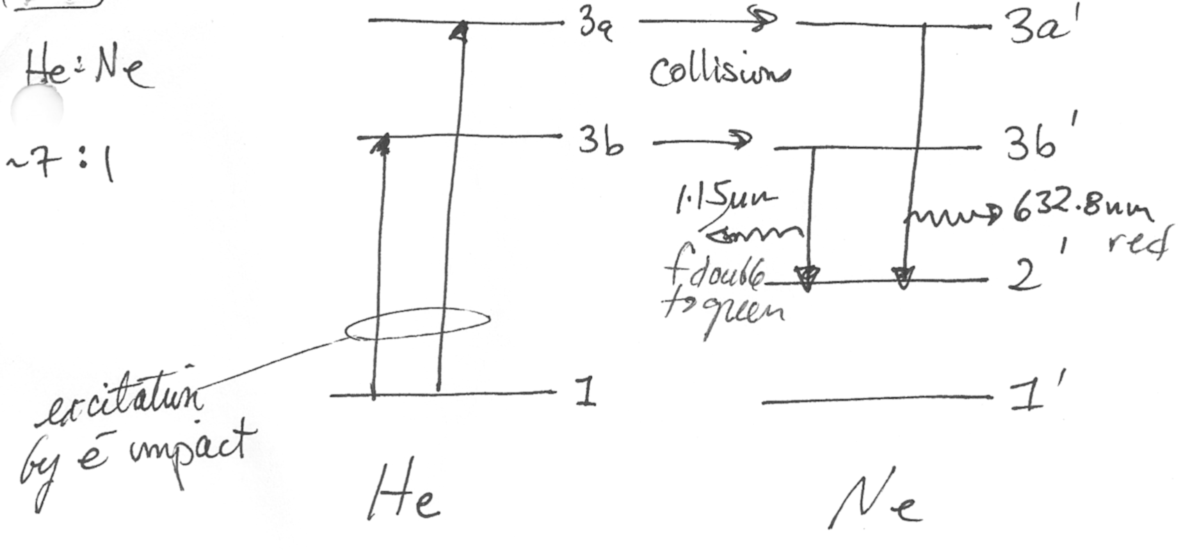



(1st laser 1960)

Gas

He:Ne

~7:1




He:Ne have matched levels!
(not an accident really!)
both noble gasses...

- excite He with accelerated ē (gas disch tube) PH2SS!
- collisions transfer E from He → Ne since levels match!
- lasing action (stim. em.) from Ne levels

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Many other types - semicon, e.g. AlGaAs-GaAs red pointer

GaN-based blue

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So? Why is this not a fancy flashlight?

- Monochromatic! best filters, ~1% or 0.1% λ spread
  laser? ~0.001% or better! to ~$10^{-12}$
  high-precision spectroscopy, timing, distance meas (via interference)

<u>Coherent</u> normal light source: short (~1m or $10^{-8}$ s), random wave trains

laser: ~$10^{-3}$ s or 1 km! nearly continuous, perfect wave

interference / diffraction on everyday scales

holograms! $\left(\underset{\Delta t}{10^{-3}\,s} \cdot \underset{v}{3\cdot10^{8}\,m/s} = \underset{\substack{100\,km \\ \sim 60\,mi}}{10^{5}\,m}!\right)$

<u>high I / low divergence</u>

$\vec{p}$ conso ⇒ same direction for all!

does not diverge unless scattered (tiny)

⇒ precision posn measurements (moon to 0.1 m)

interference w/ multiple beams

microscopy

focusing to ~$\lambda$ ... HUGE power density

precision welding, surgery

CD/DVD - measurement of $\lambda$-scale variations in reflectivity for data storage!

<u>Many, many more!</u>