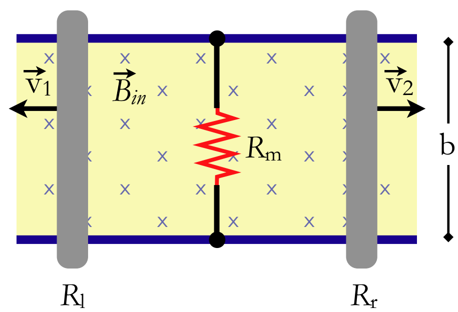
University of Alabama
Department of Physics and Astronomy

PH 102 Spring 2015

Exam 3 Solutions

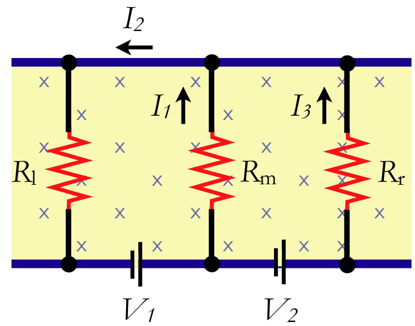
□ **1.** Two parallel rails with negligible resistance are a distance d apart and connected by a resistor R_m. The circuit also contains two metal rods, of resistance R_l and R_r, begin pulled away from the resistor at speeds of v_1 and v_2, respectively. A uniform magnetic field B is applied into the plane of the figure. Determine the current in the central resistor R_m.



Solution: The conducting rods and the fixed central resistor form two loops, a left one and a right one. As the conducting rods move, both loops have $\Delta\Phi_B/\Delta_t > 0$, i.e., the flux is increasing in both loops. The resulting induced potentials walking around each loop will try to stop the increase in flux, which means they will try to source currents that create magnetic fields working against the external field B_{in}. This means the currents must create out-of-plane magnetic fields, and so they must circulate counterclockwise in both loops.

Where to include the induced potentials? The induced potentials are *not* simply across the resistors. The induced potential shows up if one starts at some point on the loop and walks all the way around to where you started, when you reach your starting point you will have changed your potential in amount $\Delta\Phi_B/\Delta_t$. In other words, the potentials show up in the same way a battery would in the loop, as a change in potential at a single point. To analyze the situation, we can then simulate the induced voltages by placing a battery in each loop, as shown below.

The potential induced around each loop, and thus the value of each effective battery, is given by the change in flux in each loop. This is just the motionally-induced voltage we've already derived.



$$V_1 = Bbv_1 \tag{1}$$
$$V_2 = Bbv_2 \tag{2}$$

Now the problem is reduced to that of a dc circuit with two loops and two batteries. We can write two loop equations (left and right) representing conservation of energy, and a junction equation relating the three currents:

$$0 = Bbv_1 - I_1R_m - I_2R_l \tag{3}$$
$$0 = Bbv_2 - I_3R_r + I_1R_m \tag{4}$$
$$0 = I_1 - I_2 + I_3 \tag{5}$$

Solve these three equations for the three unknown currents. There are many ways to do it, but any correct method gives

$$I_1 = \frac{Bb\left(R_rv_1 - R_lv_2\right)}{R_lR_r + R_mR_r + R_lR_m} \tag{6}$$

As one might expect, whether the current in the central resistor is up or down depends on the relative resistances and velocities of the bars – the faster bar generates the larger voltage, but if it also has a higher resistance, the current may still be smaller than that generated by the other bar. Since the currents I_2 and I_3 meet head-on in the central resistor, whichever current determines the overall direction – if $R_rv_1 > R_lv_2$, the current is upward.

If you want to see it solved out, one way to do it is by Gaussian elimination (another would be Cramer's rule, or just plain brute force). I'll illustrate the Gaussian approach. First, we write the equations in matrix form:

$$\begin{bmatrix} R_m & R_l & 0 \\ R_m & 0 & -R_r \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Bbv_1 \\ -Bbv_2 \\ 0 \end{bmatrix} \tag{7}$$

A solution by Gaussian elimination is straightforward: you try to add rows together in different proportions such that one row in the coefficients matrix has only a single non-zero term left. In this case, first add R_l times the third row to the first:

$$\begin{bmatrix} R_m + R_l & R_l & 0 \\ R_m & 0 & -R_r \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Bbv_1 \\ -Bbv_2 \\ 0 \end{bmatrix} \tag{8}$$

Now add R_l/R_r times the second row to the first:

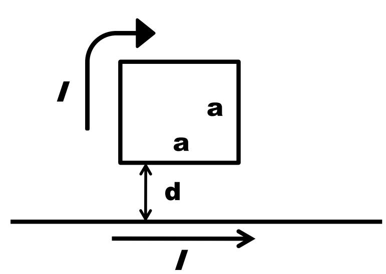
$$\begin{bmatrix} R_m + R_l + R_l R_m/R_r & 0 & 0 \\ R_m & 0 & -R_r \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} Bbv_1 \\ -Bbv_2 \\ 0 \end{bmatrix} \tag{9}$$

The top row now yields

$$I_1 = \frac{Bb\,(R_r v_1 - R_l v_2)}{R_l R_r + R_m R_r + R_l R_m} \tag{10}$$

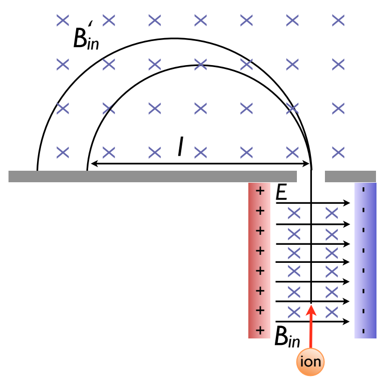
which is the solution we claimed as correct above.

□ **2.** Find the force on a square loop (side a) placed as shown below, near an infinite straight wire. Both loop and wire carry a steady current I.



□ **3.** In a mass spectrometer, a beam of ions is first made to pass through a *velocity selector* with perpendicular $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields. Here, the electric field $\vec{\mathbf{E}}$ is to the right, between parallel charged plates, and the magnetic field $\vec{\mathbf{B}}$ in the same region is into the page. The selected ions are then made to enter a region of different magnetic field $\vec{\mathbf{B}}'$, where they move in arcs of circles. The radii of these circles depend on the masses of the ions. Assume that each ion has a single charge e. Show that in terms of the given field values and the impact distance l the mass of the ion is

$$m = \frac{eBB'l}{2E}$$



Solution: Standard mass spectrometer problem from the text, just be sure to use the new field B' in the region with no electric field, and note that $l = 2r$. Otherwise, follow the derivation from the book or one you can easily find online.

□ **4.** A technician wearing a brass bracelet enclosing an area $0.00500\,\mathrm{m}^2$ places her hand in a solenoid whose magnetic field is $5.00\,\mathrm{T}$ directed perpendicular to the plane of the bracelet. The electrical resistance around the circumference of the bracelet is $0.0200\,\Omega$. A power failure causes the field to drop to $1.50\,\mathrm{T}$ in a time of $20.0\,\mathrm{ms}$. Find **(a)** the current in the bracelet, and **(b)** the power delivered to the bracelet. *Hint:* don't wear metal jewelry when working with strong magnetic fields.

Solution: Once the power goes out, the magnetic field drops by an amount $\Delta B = 5 - 1.5 = 3.5\,\mathrm{T}$ in a time $\Delta t - 0.020\,\mathrm{s}$. This means, given the fixed area of the bracelet, that the magnetic flux is changing through the bracelet:

$$\frac{\Delta\Phi_B}{\Delta t} = A\frac{\Delta B}{\Delta t} = \left(0.005\,\mathrm{m}^2\right)\left(\frac{3.5\,\mathrm{T}}{0.020\,\mathrm{s}}\right) \approx 0.875\,\mathrm{Tm}^2/\mathrm{s} = 0.875\,\mathrm{V}$$

The change in flux leads to an induced voltage across the bracelet. Since the bracelet has a single loop of wire, the induced voltage is just

$$\Delta V = -\frac{\Delta\Phi_B}{\Delta t} = -0.875\,\mathrm{V}$$

The minus sign here is not important, since the ultimate direction of current flow is not important. Given the bracelet's resistance, we can find the induced current:

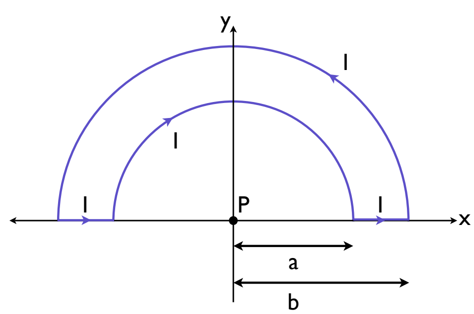
$$I = \frac{\Delta V}{R} \approx 43.75\,\text{A}$$

Finally, we can find the power from the current and voltage, or either one and the resistance:

$$\mathscr{P} = I^2 R = I\Delta V = \frac{\Delta V^2}{R} \approx 38.28\,\text{W}$$

This is an unreasonably large amount of power to be dumped into a bracelet on one's wrist, leading to an unreasonable generation of heat . . . for this reason, you will not see personnel working regularly with MRI machines wearing much jewelry. At least, they shouldn't be.

□ **5.** Find the magnetic field at point P due to the current distribution shown below. *Hint: Break the loop into segments, and use superposition.*



Solution: The easiest way to do solve this is by superposition – our odd current loop is just the same as two semicircles plus two small straight segments. We know that the magnetic field at the center of a *full* circular loop of radius r carrying a current I is

$$B = \frac{\mu_o I}{2r} \qquad \text{(loop radius r)}$$

Since the magnetic field obeys superposition, we could just as well say that our full circle is built out of two equivalent half circles! The field from each half circle, by symmetry, must be half of the total field, so the field at the center of a semicircle must simply be

$$B = \frac{\mu_o I}{4r} \qquad \text{(semicircle, radius r)}$$

In other words: a half circle gives you half the field of a full circle. Here we have two semicircular current segments contributing to the magnetic field at P: one of radius b, and one of radius a. The currents are in the opposite directions for the two loops, so their fields are in opposing directions. Based on the axes given, it is the outer loop of radius b that has its field pointing out of the page in the $\hat{\mathbf{z}}$ direction, and the inner loop of radius a in the $-\hat{\mathbf{z}}$ direction.

What about the straight bits of wire? For those segments, the direction field is zero. Since the magnetic field "circulates" around the wire, along the wire axis it must be zero. Even if it were not, by symmetry the two straight bits would have to give equal and opposite contributions and cancel each other anyway. There is no field contribution at P from the straight segments! Thus, the total field is just that due to the semicircular bits,

$$\vec{\mathbf{B}} = \frac{\mu_o I}{4b}\,\hat{\mathbf{z}} - \frac{\mu_o I}{4a}\,\hat{\mathbf{z}} = \frac{\mu_o I}{4}\left(\frac{1}{b} - \frac{1}{a}\right)\hat{\mathbf{z}}$$