PH102 Final Exam

Instructions

- 1. Answer six of the nine questions below. All problems have equal weight.
- 2. Clearly mark your which problems you have chosen using the tick box.
- 3. You are allowed 2 sheets of standard 8.5x11 in paper and a calculator.

 \Box 1. Three protons and three electrons are to be placed at the vertices of a regular octahedron of edge length **a**. We want to find the potential energy of the system, or the work required to assemble it starting with the particles infinitely far apart. There are essentially two different arrangements possible. What is the energy of each? Symbolic answer, please.



Figure 1: An octahedron. It has eight faces and six vertices.

Using the principle of superposition, we know that the potential energy of a system of charges is just the sum of the potential energies for all the unique pairs of charges. The problem is then reduced to figuring out how many different possible pairings of charges there are, and what the energy of each pairing is. The potential energy for a single pair of charges, both of magnitude q, separated by a distance d is just:

$$\mathsf{PE}_{\mathrm{pair}} = \frac{\mathsf{k}_e \mathsf{q}^2}{\mathsf{d}}$$

First, we need figure out how many pairs there are for charges arranged on the vertices of an octahedron, and for each pair, how far apart the charges are. Once we've done that, we need to figure out the two different arrangements of charges and run the numbers.

How many unique pairs of charges are there? There are not so many that we couldn't just list them by brute force – which we will do anyway to calculate the energy – but we can also calculate how many there are. In both distinct configurations, we have 6 charges, and we want to choose all possible groups of 2 charges that are not repetitions. So far as potential energy is concerned, the pair (2, 1) is the same as (1, 2). Pairings like this are known as combinations, as opposed to *permutations* where (1, 2) and (2, 1) are *not* the same. Calculating the number of possible combinations is done like this:ⁱ

ways of choosing pairs from six charges
$$= \binom{6}{2} = {}^{6}C_{2} = \frac{6!}{2!(6-2)!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 15$$

We can verify this by simply enumerating all the possible pairings. Label the charges at each vertex in some fashion, such as this:



We have six charges at six vertices, and thus ${}_{6}C_{2} = \frac{6!}{2!4!} = 15$ unique pairings of charges. Namely,

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q_1 q_2, q_1 q_3, q_1 q_4, q_1 q_5, q_1 q_6
q_2 q_3, q_2 q_4, q_2 q_5, q_2 q_6
q_3 q_4, q_3 q_5, q_3 q_6
q_4 q_5, q_4 q_6
q_5 q_6
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Here all the q_i have the same magnitude, the labels are just to keep things straight. At a given vertex, all four nearest-neighbor vertices are at distance a, while the single "next-nearest neighbor" is at a distance $a\sqrt{2}$. This means that there are three *pairs* charges which are separated by a distance $a\sqrt{2}$, and the other twelve pairings are at a distance a. We have highlighted the $a\sqrt{2}$ pairings above. How can we find two different arrangements? Since there are an odd number of next-nearest neighbor pairings, the first suspicion is that the difference between the two arrangements will be in next-nearest neighbor pairings. If you experiment for a while, the two different arrangements are these:

Now we need only add up the potential energies of all possible pairs of charges. All the nearestneighbor pairs will have the same energy, viz.,

$$|\mathbf{U}_{nn}| = \frac{\mathbf{kq}^2}{\mathbf{a}} \tag{1}$$

ⁱA nice discussion of combinations and permutations is here: http://www.themathpage.com/aPreCalc/permutations-combinations.htm



All the next-nearest neighbor pairs will have

$$|\mathbf{U}_{nnn}| = \frac{\mathbf{k}\mathbf{q}^2}{\mathbf{a}\sqrt{2}} \tag{2}$$

For the first arrangement we have 12 nearest-neighbor pairs: eight of them are +- pairings, and four of them are ++ or -- pairs. We have three next-nearest neighbor pairs, two ++ or --, and one +-. Thus, the total energy must be

$$\mathbf{U}_{\mathsf{A}} = 8\left[\frac{-\mathsf{k}\mathsf{q}^2}{\mathsf{a}}\right] + 4\left[\frac{\mathsf{k}\mathsf{q}^2}{\mathsf{a}}\right] + 2\left[\frac{\mathsf{k}\mathsf{q}^2}{\mathsf{a}\sqrt{2}}\right] + 1\left[\frac{-\mathsf{k}\mathsf{q}^2}{\mathsf{a}\sqrt{2}}\right] = \frac{\mathsf{k}\mathsf{q}^2}{\mathsf{a}}\left[\frac{1}{\sqrt{2}} - 4\right] = \left[\frac{1}{\sqrt{2}} - 4\right] |\mathbf{U}_{\mathsf{n}\mathsf{n}}| \approx -3.29|\mathbf{U}_{\mathsf{n}\mathsf{n}}|$$
(3)

For the second arrangement, of the 12 nearest-neighbor pairs we have six +- pairs and six ++ or -- pairs, and thus the total energy of nearest-neighbor pairs will be zero. We are left with only the next-nearest neighbor terms, and for this arrangement, all three are +- pairs. Thus,

$$U_{\rm B} = -3 \frac{kq^2}{a\sqrt{2}} = \frac{3}{\sqrt{2}} |U_{nn}| \approx -2.12 |U_{nn}| \tag{4}$$

Thus, $U_A < U_B$, and the first lattice is more stable, owing to its lower nearest-neighbor energy. Though the second lattice has a smaller next-nearest neighbor energy, there are fewer next-nearest neighbor pairs, and their energy is smaller than the nearest neighbor pairs. Usually, minimizing the nearest-neighbor energy gives the most stable crystal, simply because the potential is decreasing with distance.

^{\Box} **2.** An interstellar dust grain, roughly spherical with a radius of 3×10^{-7} m, has acquired a negative charge such that its electric potential is -0.15 Volts.

- (a) How many extra electrons has it picked up?
- (b) What is the strength of the electric field at its surface?

If it is spherical, Gauss' law tells us that we may treat it as a point charge (so long as we are outside the dust grain, anyway). The excess charge must therefore be equivalent to a point charge which at a distance $d = 3 \times 10^{-7}$ m creates a potential of -0.15 Volts. If there are n excess electrons on the dust grain, the net charge is $q_{net} = -ne$. Thus,

$$-0.15 \,\mathrm{V} = \frac{\mathrm{kq}_{\mathrm{net}}}{\mathrm{d}} = \frac{-\mathrm{kn}e}{3 \times 10^{-7} \,\mathrm{m}}$$
$$n \approx 31 \,\mathrm{electrons} \tag{5}$$

Here we rounded to the nearest integer for n. The same point charge would produce an electric field at a distance of 3×10^{-7} m of

$$E = \frac{-kne}{(3 \times 10^{-7} \,\mathrm{m})^2} \approx 5 \times 10^5 \,\mathrm{V/m}$$
(6)

 \square 3. A 50 kV direct-current power line consists of two conductors 2 m apart. When this line is transmitting a power of 10 MW, how strong is the magnetic field halfway between the conductors?

 \Box 4. A spaceship traveling at 0.70c away from the Earth launches a projectile of muzzle speed 0.90c (relative to the spaceship). What is the speed of the projectile relative to Earth if it is launched in the forward direction? In the backward direction?

 \Box 5. Two positively charged particles separated by a distance d, each with charge q and mass m, are initially moving at the same speed ν in opposite directions perpendicular to the line joining them. A magnetic field applied perpendicularly to the plane of the page will bend the paths of the particles into circles. What strength of magnetic field is necessary to make them collide head-on midway between the two starting points? (Ignore the electrical forces between the charges.)

 \square 6. The bottom half of a beaker of depth 20 cm is filled with water (n=1.33) and the top half is filled with oil (n=1.48). If you look into this beaker from above, how far below the upper surface of the oil does the bottom of the beaker seem to be?

We must account for refraction in both materials. In general, looking into a material of index n_2 from a material of index n_1 , the apparent depth is

$$\mathbf{d}_{\mathrm{app}} = \mathbf{d}_{\mathrm{real}} \frac{\mathbf{n}_1}{\mathbf{n}_2} \tag{7}$$

Here we have air, water, and oil, which we will give indices n_a , n_w , and n_o , respectively. Looking down through the oil, which has an actual depth $d_{real} = 10$ cm, we would say its boundary with the water would be at a depth

$$d_{\rm oil} = d_{\rm real} \frac{n_a}{n_o} \tag{8}$$

What about looking through the water layer below? Same thing, the light will be refracted, but

now in an amount dictated by the index of the water. We don't need to worry about the oil since the rays of interest both enter and exit the oil, the apparent depth of the *water* doesn't depend on the oil being there or not.

$$\mathbf{d}_{\mathrm{water}} = \mathbf{d}_{\mathrm{real}} \frac{\mathbf{n}_{\mathrm{a}}}{\mathbf{n}_{w}} \tag{9}$$

The total apparent depth of the beaker is then

$$d_{\rm app} = d_{\rm water} + d_{\rm oil} = d_{\rm real} \left(\frac{n_a}{n_w} + \frac{n_a}{n_o} \right) \approx 14.2 \,\rm cm \tag{10}$$

□ 7. A particle of charge q and mass m, moving with a constant speed v perpendicular to a constant magnetic field B follows a circular path. If the angular momentum is quantized so that $|\vec{L}| = mvr = n\hbar$, determine the allowed radii for the particle in terms of the preceding quantities.

□ 8. The circuit at right is known as a *Wheatstone Bridge*, and it is a useful circuit for measuring small changes in resistance. Perhaps you can figure out why. Three of the four branches on our bridge have identical resistance R, but the fourth has a slightly different resistance, ΔR more than the other branches, such that its total resistance is $R + \Delta R$.



Wheatstone Bridge

In terms of the source voltage V_s , base resistance R and change in resistance ΔR , what is the potential difference between points a and b? You may assume the voltage source and wires are perfect (no internal resistance and no voltage drop, respectively).

Remember that ideal voltmeters draw no current, so the meter in the center of the bridge doesn't really *do* anything. If we label the nodes on the bridge a-d, as shown in the figure at right, the voltmeter simply tells us the potential difference between points d and b, ΔV_{db} . Knowing that, we will simply leave it out of our diagram to make things a bit more clear.

Looking more carefully at the bridge, we notice that it is nothing more than two sets of series resistors, connected in parallel with each other. This immediately means that the voltage drop across the left side of the bridge, following nodes $a \rightarrow d \rightarrow c$, must be the same as the voltage drop across the right side of the bridge, following nodes $a \rightarrow b \rightarrow c$ – both are ΔV_{ac} , and both must be the same as the source voltage: $\Delta V_{ac} = V_s$. If we can find the current in each resistor, then with the known source potential difference we will know the voltage at any point in the circuit we like, and finding ΔV_{db} is no problem.

Let the current from the source V_s be I. This current I leaving the source will at node a split in to separate currents I_1 and I_2 ; conservation of charge requires $I = I_1 + I_2$. At node c, the currents recombine into I. On the leftmost branch of the bridge, the current I_1 creates a voltage drop I_1R across each resistor. Similarly, on the rightmost branch of the bridge, each the resistor R has a voltage drop I_2R and the lower resistor has a voltage drop $I_2(R + \delta R)$. Equating the total voltage drop on each branch of the bridge:

$$\begin{split} V_s &= I_1 R + I_1 R = I_2 R + I_2 \left(R + \delta R \right) \\ \Longrightarrow \quad I_1 &= \frac{V_s}{2R} \\ I_2 &= \frac{V_s}{2R + \delta R} \end{split}$$

Now that we know the currents in terms of known quantities, we can find ΔV_{db} by "walking" from point d to point b and summing the changes in potential difference. Starting at node d, we move toward node a *against* the current I₁, which means we *gain* a potential difference I₁R. Moving from node a to node b, we move *with* the current I₂, which means we *lose* a potential difference I₂R. Thus, the total potential difference between points d and b must be

$$\begin{split} \Delta V_{db} &= I_1 R - I_2 R = R \left(I_1 + I_2 \right) = R \left(\frac{V_s}{2R} - \frac{V_s}{2R + \delta R} \right) \\ \Delta V_{db} &= V_s \left(\frac{1}{2} - \frac{R}{R + \delta R} \right) = V_s \left(\frac{\delta R}{4R + 2\delta R} \right) \end{split}$$

If the change in resistance δR is small compared to R ($\delta R \ll R$), the term in the denominator can be approximated $4R + \delta R \approx 4R$, and we have

$$\Delta V_{db} = V_s \frac{\delta R}{4R} \qquad (\delta R \ll R)$$

Thus, for small changes in resistance, the voltage measured across the bridge is directly proportional to the change in resistance, which is the basic utility of this circuit: it allows one to measure small changes on top of a large 'base' resistance. Fundamentally, it is a *difference* measurement, meaning that one directly measures *changes* in the quantity of interest, rather than measuring the whole thing and trying to uncover subtle changes. This behavior is very useful for, e.g., strain gauges, temperature sensors, and many other devices.

Let z = 0 at the intersection of the plane of the bottom coil and the z axis. The field from the bottom coil at an arbitrary point a distance z along the axis due to the bottom coil is just the quantity given above. At a position z, since the separation of the coils is R, we are a distance R - z from the upper coil. We need only replace z with R - z in the expression above to find the field from the upper coil at a distance z < R from the bottom coil. Since the currents are in the same directions for both coils, the magnetic fields are in the same direction, and we may just add them together:

 \square 9. A Helmholtz coil consists of two identical circular coils separated by a distance equal to their radius R, as shown at right. Each carries current I in the same direction. Find the field at any point along the axis between the two coils (the z axis in the figure). *Hint:* The field from a single loop of radius R a distance z along the axis is:

$$\mathsf{B} = \frac{\mu_0 \mathrm{I}}{2} \frac{\mathsf{R}^2}{\left(z^2 + \mathsf{R}^2\right)^{3/2}} \quad \text{(single loop)}$$



Helmholtz Coil

$$B_{\text{tot}} = B_{\text{lower}} + B_{\text{upper}} = \frac{\mu_o I}{2} \frac{R^2}{(z^2 + R^2)^{3/2}} + \frac{\mu_o I}{2} \frac{R^2}{\left[(R - z)^2 + R^2\right]^{3/2}}$$

Electric Potential:

Quadratic formula:

$$0 = ax^{2} + bx^{2} + c \Longrightarrow x = \frac{-b \pm \sqrt{b^{2} - 4ac}}{2a}$$

Basic Equations:

$$\vec{F}_{net} = m\vec{a}$$
 Newton's Second Law
 $\vec{F}_{centr} = -\frac{mv^2}{r}\hat{r}$ Centripetal

Current:

$$I = \frac{\Delta Q}{\Delta t} = nqAv_d$$

$$J = \frac{I}{A} = nqv_d$$

$$v_d = \frac{-e\tau}{m}E \quad \tau = \text{scattering time}$$

$$\rho = \frac{m}{ne^2\tau}$$

$$\Delta V = \frac{\rho I}{A}I = RI$$

$$R = \frac{\Delta V}{I} = \frac{\rho I}{A}$$

$$\mathscr{P} = E \cdot \Delta t = I\Delta V = I^2 R = \frac{[\Delta V]^2}{R} \text{ power}$$

Ohm:

$$\begin{aligned} \mathscr{E} &= hf = \frac{hc}{\lambda} \\ n &= \frac{\text{speed of light in vacuum}}{\text{speed of light in a medium}} = \frac{c}{\nu} \\ \frac{\lambda_1}{\lambda_2} &= \frac{\nu_1}{\nu_2} = \frac{c/n_1}{c/n_2} = \frac{n_2}{n_1} \quad \text{refraction} \\ n_1 \sin \theta_1 &= n_2 \sin \theta_2 \quad \text{Snell's refraction} \\ \lambda f &= c \\ M &= \frac{h'}{h} = -\frac{q}{p} \\ \frac{1}{f} &= \frac{1}{p} + \frac{1}{q} = \frac{2}{R} \quad \text{mirror } \& \text{ lens} \\ \frac{n_1}{p} + \frac{n_2}{q} &= \frac{n_2 - n_1}{R} \quad \text{spherical refracting} \\ q &= -\frac{n_2}{n_1}p \quad \text{flat refracting} \\ \frac{1}{f} &= \left(\frac{n_2 - n_1}{n_1}\right) \left[\frac{1}{R_1} - \frac{1}{R_2}\right] \quad \text{lensmaker's} \end{aligned}$$

Electric Force & Field

$$\begin{split} \vec{F}_{e,12} &= q\vec{E}_{12} = \frac{k_e q_1 q_2}{r_{12}^2} \hat{r}_{12} \\ \vec{E} &= k_e \frac{|q|}{r^2} \\ \Phi_E &= |\vec{E}| A \cos \theta_{EA} = \frac{Q_{inside}}{\varepsilon_0} \\ \Delta PE &= -W = -q |\vec{E}| |\Delta \vec{x}| \cos \theta = -q E_x \Delta x \\ \uparrow \text{ constant E field} \end{split}$$

Capacitors:

$$\begin{array}{rcl} Q_{\rm capacitor} & = & C \Delta V \\ C_{\rm parallel \ plate} & = & \displaystyle \frac{\varepsilon_0 A}{d} \\ E_{\rm capacitor} & = & \displaystyle \frac{1}{2} Q \Delta V = \displaystyle \frac{Q^2}{2C} \\ C_{\rm eq, \ par} & = & \displaystyle C_1 + C_2 \\ C_{\rm eq, \ series} & = & \displaystyle \frac{C_1 C_2}{C_1 + C_2} \\ C_{\rm with \ dielectric} & = & \displaystyle \kappa C_{\rm without} \end{array}$$

Resistors:

$$\begin{split} \mathrm{I}_{\mathrm{V}\ \mathrm{source}} &=& \frac{\Delta \mathrm{V}_{\mathrm{rated}}}{\mathrm{R}+\mathrm{r}} \\ \Delta \mathrm{V}_{\mathrm{V}\ \mathrm{source}} &=& \Delta \mathrm{V}_{\mathrm{rated}} \frac{\mathrm{R}}{\mathrm{r}+\mathrm{R}} \\ \mathrm{I}_{\mathrm{I}\ \mathrm{source}} &=& \mathrm{I}_{\mathrm{rated}} \frac{\mathrm{r}}{\mathrm{r}+\mathrm{R}} \\ \mathrm{R}_{\mathrm{eq},\ \mathrm{series}} &=& \mathrm{R}_{1}+\mathrm{R}_{2} \\ \frac{1}{\mathrm{R}_{\mathrm{eq},\ \mathrm{par}}} &=& \frac{1}{\mathrm{R}_{1}} + \frac{1}{\mathrm{R}_{2}} \end{split}$$

RC circuits

		- [t.	(π]
$Q_{C}(t)$	=	$Q_0 \left[1 - e^{-t} \right]$	'] charging
$Q_{C}(t)$	=	$Q_0 e^{-t/\tau}$	discharging
Q(t)	=	$C\Delta V(t)$	
τ	=	RC	

Vectors:

$$\begin{split} \vec{F}| &= \sqrt{F_x^2 + F_y^2} \ \text{magnitude} \\ \theta &= \ \tan^{-1}\left[\frac{F_y}{F_x}\right] \ \text{direction} \end{split}$$

Induction:

ac Circuits

τ	=	L/R	RL circuit
τ	=	RC	RC circuit
x _c	=	$\frac{1}{2\pi fC}$	"resistance" of a capacitor for ac
XL	=	$2\pi fL$	"resistance" of an inductor for ac
$\omega_{ m cutoff}$	=	$\frac{1}{\tau} = 2\pi$	f

Relativity

$$\begin{split} \gamma &= \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\ \Delta t'_{moving} &= \gamma \Delta t_{stationary} = \gamma \Delta t_p \\ L'_{moving} &= \frac{L_{stationary}}{\gamma} = \gamma \left(x - \nu t \right) \\ \Delta t' &= t'_1 - t'_2 = \gamma \left(\Delta t - \frac{\nu \Delta x}{c^2} \right) \\ \nu_{obj} &= \frac{\nu + \nu'_{obj}}{1 + \frac{\nu \nu'_{obj}}{c^2}} \qquad \nu'_{obj} = \frac{\nu_{obj} - \nu}{1 - \frac{\nu \nu_{obj}}{c^2}} \\ KE &= (\gamma - 1)mc^2 \\ E_{rest} &= mc^2 \qquad p = \gamma m\nu \\ E^2 &= p^2c^2 + m^2c^4 \end{split}$$

Right-hand rule #1

- 1. Point the fingers of your right hand along the direction of $\vec{\nu}.$
- 2. Point your thumb in the direction of $\vec{B}\,.$
- 3. The magnetic force on a + charge points out from the back of your hand.

Right-hand rule #2:

Point your thumb along the direction of the current (magnetic field). Your fingers naturally curl around the direction the magnetic field (current) circulates.

Unit	\mathbf{Symbol}	equivalent to
newton	Ν	$kg \cdot m/s^2$
joule	J	$kg \cdot m^2/s^2 = N \cdot m$
watt	W	$J/s=m^2 \cdot kg/s^3$
coulomb	\mathbf{C}	A·s
amp	А	C/s
volt	V	$W/A = m^2 \cdot kg / \cdot s^3 \cdot A$
farad	\mathbf{F}	$\mathrm{C/V}{=}\mathrm{A}^2{\cdot}\mathrm{s}^4/\mathrm{m}^2{\cdot}\mathrm{kg}$
ohm	Ω	$V/A = m^2 \cdot kg/s^3 \cdot A^2$
tesla	Т	$Wb/m^2 = kg/s^2 \cdot A$
electron volt	eV	$1.6\times10^{-19}\mathrm{J}$
-	$1\mathrm{T}\cdot\mathrm{m/A}$	$1\mathrm{N/A^2}$
-	$1\mathrm{T}\cdot\mathrm{m}^2$	$1\mathrm{V}\cdot\mathrm{s}$
-	$1\mathrm{N/C}$	$1\mathrm{V/m}$
	,	,

Power	Prefix	Abbreviation
10^{-12}	pico	р
10^{-9}	nano	n
10^{-6}	micro	μ
10^{-3}	milli	m
10^{-2}	centi	с
10^{3}	kilo	k
10^{6}	mega	М
10^{9}	giga	G
10^{12}	tera	Т